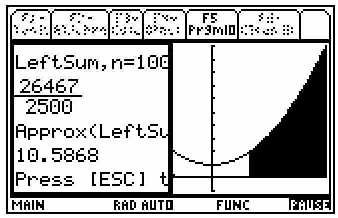
Math 163
Lab 13

AREA UNDER A CURVE

There are two fundamental problems that The Calculus allows us to solve:

- Find the slope of a curve at any point on the curve (where the curve is differentiable), and
- Find the area under a curve; i.e., the area between the x-axis and the graph of a function.

The shaded area of the graph below shows the area between the x-axis and the graph of $y = x^2 + 1$, from $x = 1$ to $x = 3$. The approximate area is shown on the left side of the screen: 10.5868. The actual area (which The Calculus enables us to find) is $10\frac{2}{3} \approx 10.6666\ldots$



Formulas you have worked with in Geometry give areas of polygonal regions, or circles, but not regions bordered partially or totally by curves of other types. This lab activity will show you how mathematicians begin to deal with such problems.

You will have to download a program from the instructor's calculators. This will only work if you have the most recent version of the calculator operating system. Here is how to check it:

1. Go to the **HOME** screen, and press **[F1] :TOOLS**.
2. Choose **Option A: About**
3. Look carefully at the third line of information. If it reads:
   Version 2.05 …., (or later) you should be able to download the software application.
   Otherwise, work with someone who ***is*** able to download the software.

(If you have any version earlier (lower) than version <u>2.05</u>, you will NOT be able to run the program successfully. Your calculator will lock up, and it will take an act of the TI-89 supreme beings to unlock it. I have fortunately been entrusted with this magical process - see me during office hours.)

To download the application **CalcTools**:

Plug the cable firmly, but gently, into both calculators

"Receiver" – press **[2nd] [VAR-LINK], [F3]: Link_**, and select **Choice 2: Receive.**

The message: "**VAR-LINK: WAITING TO RECEIVE**" should appear on the status line.
Wait for sender to complete their tasks.

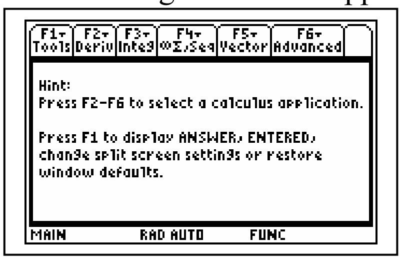
"Sender" – press **[2nd] [VAR-LINK]**, then **[2nd] [F7]: FlashApp**

Move the cursor up or down to highlight the line **Calculus Tools,**
then press **[F4] (√)_** to select it.
Now press **[F3] : Link_**, and select **choice 1: Send to TI-89/92 Plus**.
This will transfer the program.

This is a lengthy program, and will take about 20 seconds to be transferred. A bar graph at the bottom of the screen will display the progress of the link operation.

Before you run the program, change the window coordinates to [-1, 3] X [-1, 12]. The window cannot be set from within **Calctools**.

To run the **Calculus Tools**, press **[APPS] - 1**: **FlashApps** , then choose **Calculus Tools**. The following screen will appear:



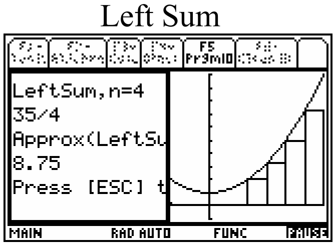
We will be working with option **[F3]: Integ** . Choose it and a menu appears:

The first five options give five different techniques for finding the area under a curve. Use the following steps to use any of them.:

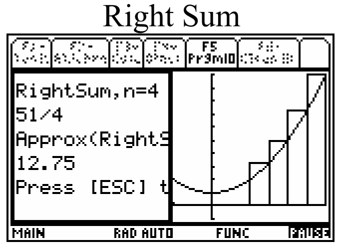
Divide the region whose area you wish to find into a series of vertical "slices" (known as partitions). Usually we take partitions of equal width. Then, in the first three options: Left Sum, Right, and Midpoint Rule, we construct rectangles using the Left edge, Right edge, or Midpoint of the base of each partition at heights:

- Press **[2nd] – [QUIT]** to return to the **HOME** screen
- Set the Window: **[-1, 3]×[-1, 10], xscl=1; yscl=1; xRes=1**
- Press **[APPS] – choice 1: FlashApps-CalculusTools – [F3]: Integ**, and choose **1: Left Sum**;
- Enter **x^2+1** for **f(x), Lower bound 1, upper bound 3, Number of partitions 4; [ENTER]-[ENTER]**
- Repeat for **Right Sum** and **Midpoint Rule.**

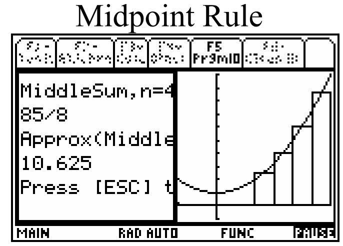
Left Sum



Right Sum

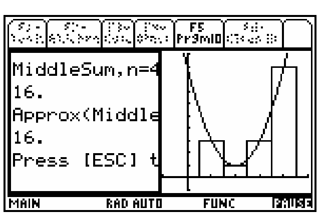


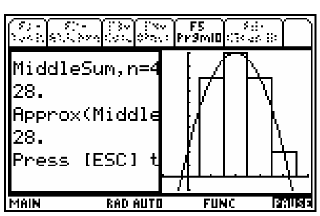
Midpoint Rule



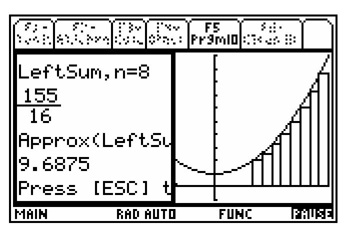
Notice how the Left Sum "misses" some of the area under the curve. It would enclose extra area if the function were decreasing. The right Sum encloses some "extra" area. It would miss some area if the function were decreasing. The Midpoint Rule has some extra area, as well as missing some area.

At a local minimum, it might just "miss" some area, and at a local max in might enclose some "extra" area.



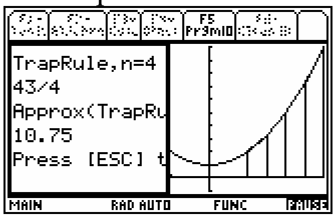


Note that with more partitions the sum of the areas of the rectangles gets closer to the real area under the curve (compare to left endpt. approx. above)

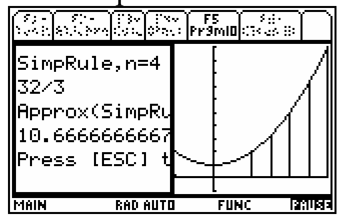


The Trapezoidal Rule and Simpson's Rule are two more ways of approximating the area:

Trapezoidal Rule

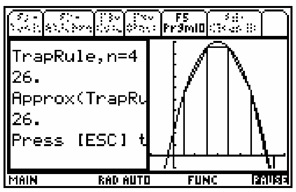


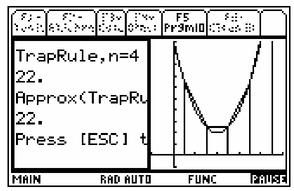
Simpson's Rule



The trapezoidal rule connects upper endpoints of the intervals with line segments, and joins them to form trapezoids. Simpson's Rule uses a "weighted average of the midpoint and trapezoidal approximations." Notice that in this case, Simpson's Rule gives the exact value. You will study this more in Calculus II.

With the Trapezoidal Rule, the "extra" or "missed" area depends on the concavity of the graph.





With only 4 intervals or partitions for an interval, we should expect that the sum of the areas of the rectangles or trapezoids would be only a rough approximation of the actual area under the curve. However, if we allow the number of intervals (partitions) to increase, the area enclosed should get closer to the actual area under the curve.

Using the same function: $y = x^2 + 1$ as before, from $x = 1$ to $x = 3$. You complete the table, using the program. (The actual area is $10\frac{2}{3}$):

Technique	Intervals	Area
Left Sum	4	8.75
	16	
	100	
Right Sum	4	12.75
	16	
	100	
Midpoint Rule	4	10.625
	16	
	100	
Trapezoidal Rule	4	10.75
	16	
	100	

We have noticed that as the number of intervals or partitions (of equal width) increases, the sum of the areas the rectangles (or trapezoids) enclose gets closer to the actual area under the curve. Hence, the area under the curve becomes the *limit* of the sum of the areas of the rectangles or trapezoids as the number of partitions, n, approaches infinity:

$$\text{Area under the curve} = \lim_{n\to\infty}(\text{sum of the } n \text{ rectangles or trapezoids})$$

Symbolically, this is written as a "***DEFINITE INTEGRAL:***"

Right border

Left border

$$\int_1^3 f(x)\ dx$$

function

Width of each sub-interval. When this approaches zero, the value obtained approaches the "true" area under the curve

If the left border is $x = a$, and the right border is $x = b$, and there are n intervals of equal width, then the definite integral is written as

area of the partition

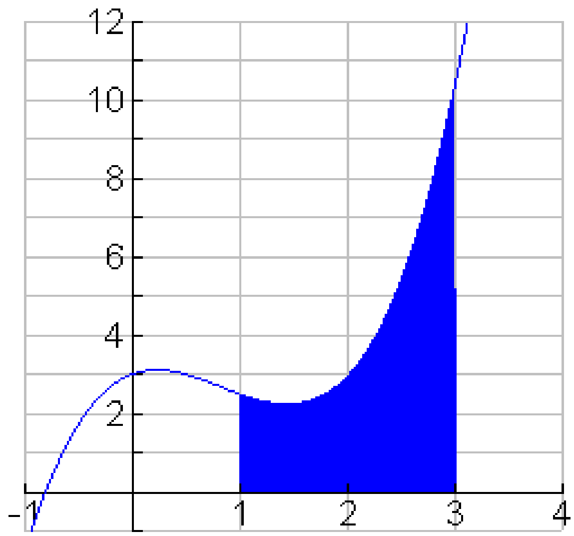
$$\int_a^b f(x)\ dx \text{ and is equal to } \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{(b-a)}{n}\right)\left(f(x_i)\right)\right)$$

The sum of the n partitions

width of each partition *height of each partition*

The following shows the area under the graph of $f(x) = x^3 - 2.5x^2 + x + 3$ from $x = 1$ to $x = 3$.

The actual value of $\int_1^3 (x^3 - 2.5x^2 + x + 3)dx = 8\frac{1}{3}$.

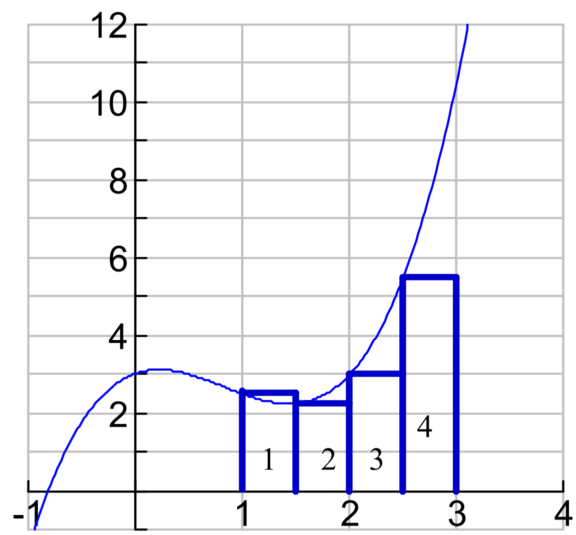


Values of *f* to use in the homework problems:

$f(1) = 2.5$	$f(1.25) = 2.296875$
$f(1.5) = 2.25$	$f(1.75) = 2.453125$
$f(2) = 3$	$f(2.25) = 3.984375$
$f(2.5) = 5.5$	$f(2.75) = 7.640625$
$f(3) = 10.5$	

Here is the same function, with 4 sub-intervals, using left endpoints.

(Area for this partition is worked out below.)



We use the function values at the left.

Area of rectangle = $b \times h$

rectangle 1 = $.5 \times f(1)$ = $.5 \times 2.5$ = 1.250

rectangle 2 = $.5 \times f(1.5)$ = $.5 \times 2.25$ = 1.125

rectangle 3 = $.5 \times f(2)$ = $.5 \times 3$ = 1.500

rectangle 4 = $.5 \times f(2.5)$ = $.5 \times 5.5$ = <u>2.750</u>

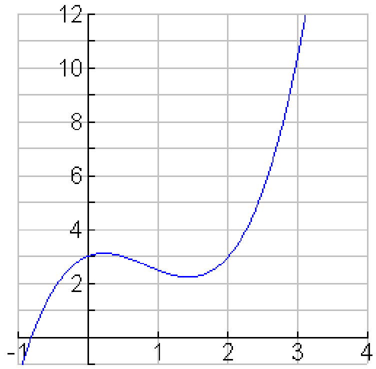
Total = 6.625

1. Using a straightedge, draw the partitioning and find the area under the curve from $x = 1$ to $x = 3$ using the right endpoint rule with 4 intervals. Use **calctools** only as a check of your work.

Use the appropriate values for $f(1), f(1.25), \ldots, f(3)$ as shown in the table at the lower left of the previous page (Show your work in the same form as demonstrated in the example shown on the previous page).

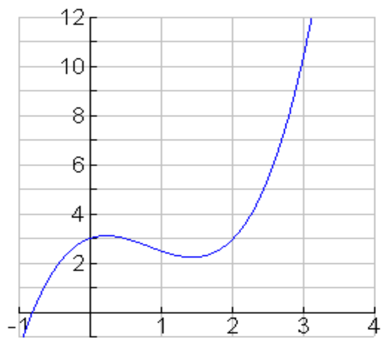
Show your work and answer in the space at the right.

Right Endpoint Rule



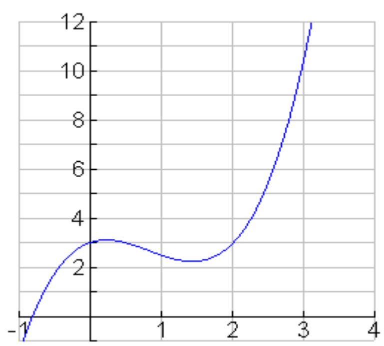
Right Endpoint Area estimate________________

2. Using a straightedge, draw the partitioning and find the area under the curve from $x = 1$ to $x = 3$ using the midpoint rule with 4 intervals. Use calctools to find the area using 4 partitions.



Midpoint Rule Area estimate__________

3. Using a straightedge, draw the partitioning and find the area under the curve from $x = 1$ to $x = 3$ using the trapezoidal rule with 4 intervals. Use calctools to find the area using 4 partitions.



Trapezoidal Rule Area estimate__________