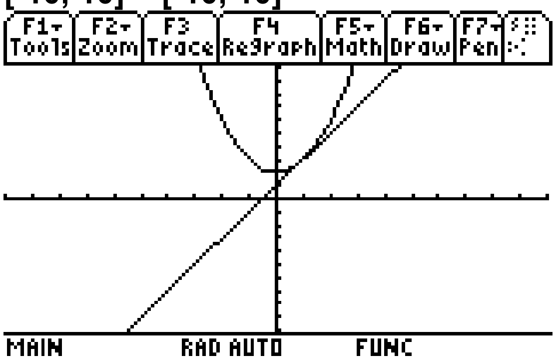
Math 163
Lab 12

Linear and Quadratic Approximations;   Taylor Polynomials

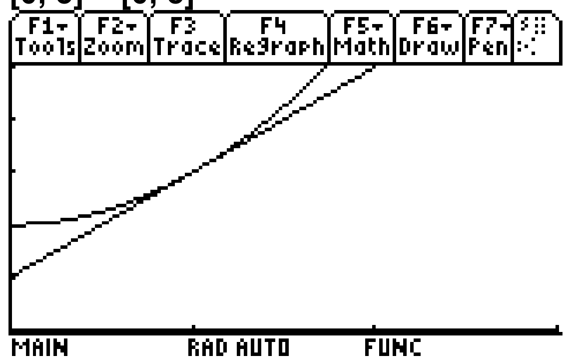
Linear Approximations

We have seen that the tangent line to the graph of $f(x)$ at $x = a$ is very close to the graph of $f(x)$ near the point $(a, f(a))$. For example, the tangent to the graph of $f(x) = x^2 + 2$ at the point (1, 3) can be found to be $y = 2x + 1$. If we graph both the parabola and the tangent line, then zoom in around the point (1, 3), it is easy to see that as we look at smaller and smaller neighborhoods around (1, 3), the difference between the parabola and tangent line becomes smaller and smaller.

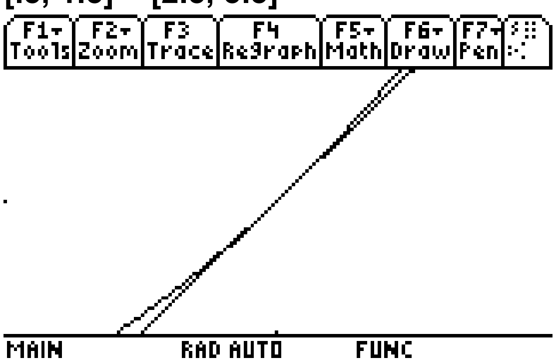
[-10, 10] × [-10, 10]



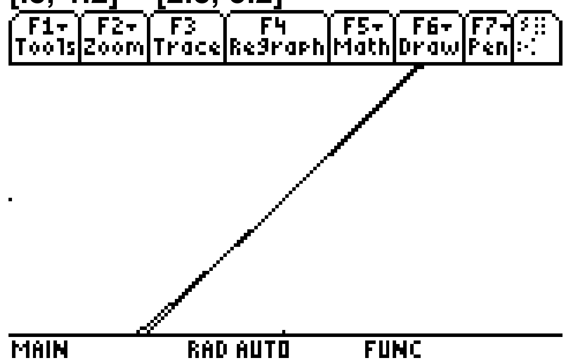
[0, 3] × [0, 5]



[.5, 1.5] × [2.5, 3.5]



[.8, 1.2] × [2.8, 3.2]



We write the equation of the tangent to $y = f(x)$ at $x = a$ using the point-slope form $y - y_1 = m(x - x_1)$ together with the point $(a, f(a))$ and slope $m = f'(a)$:

$$y - f(a) = f'(a)(x - a), \quad or$$

$$y = f(a) + f'(a)(x - a) \qquad (*)$$

We call the function (*) ***the local linearization*** **or** ***linear approximation*** of $f(x)$ at $x = a$, and denote it by

$$L_a(x) = f(a) + f'(a)(x - a)$$

Another Example:

Find the local linearization of $f(x) = \sin x$ at $a = \pi$. Graph f and its local linearization on the same axes. Then zoom in several times to see this behavior. Finally, use the linearization formula determined to find sin 3 and sin 3.1, and compare to the values obtained to those the calculator gives from the command line.

Using the formula for local linearization: $L_a(x) = y = f(a) + f'(a)(x-a)$

Here,

$$a = \pi \text{, and } f(x) = \sin x. \qquad \therefore f'(x) = \cos x$$

Consequently:

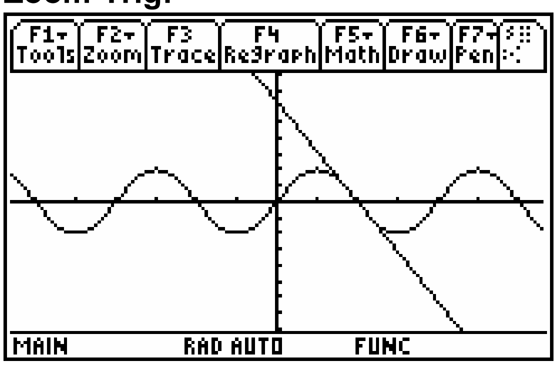
$$f(a) = \sin a = \sin \pi = 0 \quad \textit{and} \quad f'(a) = \cos a = \cos \pi = -1$$

Substituting π for a and using the values above, we obtain

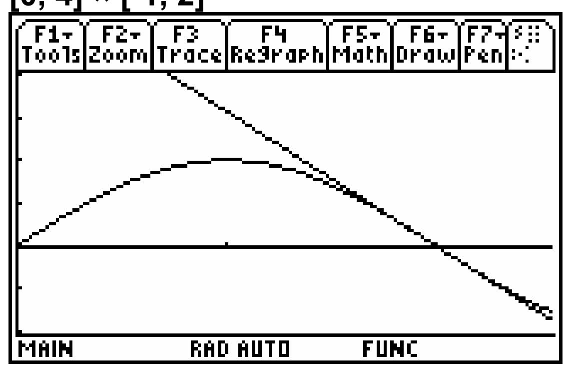
$$L_\pi(x) = f(\pi) + f'(\pi)(x - \pi)$$
$$L_\pi(x) = 0 + (-1)(x-\pi)$$
$$L_\pi(x) = -x + \pi$$

This function is the local linearization of $f(x) = \sin x$ at $a = \pi$

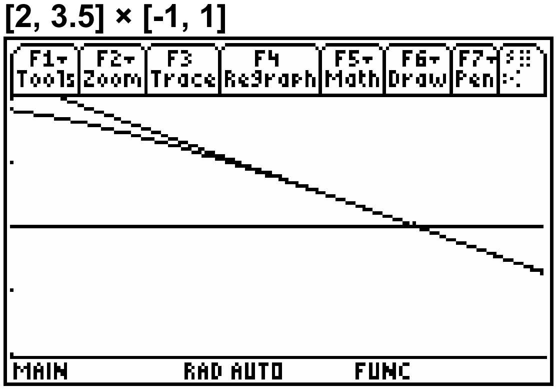
Zoom-Trig:



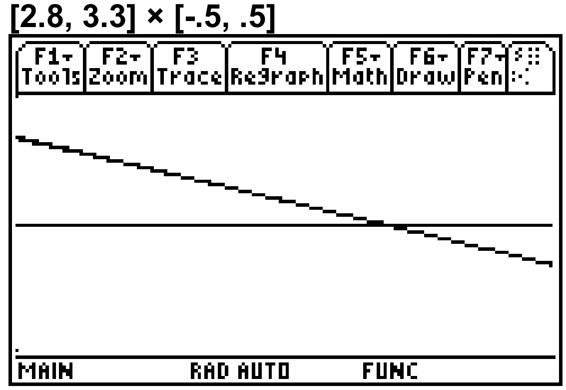
[0, 4] × [-1, 2]



[2, 3.5] × [-1, 1]



[2.8, 3.3] × [-.5, .5]



In the era when calculators and even trig tables were not available, the advantage of having this local approximation of the sin function for values of x close to π is that we can get a good approximation of $\sin x$ by finding the value of its local linearization, which is possible to do by hand:

Using local linearization:	$\sin 3 \approx -3 + \pi \approx .14159$	$\sin 3.1 \approx -3.1 + \pi \approx .04159$
Using the calculator:	$\sin 3 \approx .14112$	$\sin 3.1 \approx .04158$

Quadratic Approximations; The Quadratic Taylor Polynomial

Using the derivative for the slope simply guarantees that the tangent line, or local linearization of $f(x)$ "goes in the same direction" as the graph of $f(x)$ does at $x = a$. We can make our approximation "curve the same way that f does" if we ***also*** include a second derivative term in our approximation. This is called the **Quadratic Approximation**.

$Q_a(x)$, the *quadratic approximation* of $f(x)$ at $x = a$ is the *quadratic function* that most closely resembles the graph of $f(x)$ near the point $(a, f(a))$.

We generate this approximation by finding a quadratic function $Q_a(x)$ satisfying:

$(at\ a, the\ functions\ are\ equal):$ 1) $Q_a(a) = f(a)$

$(at\ a, the\ slopes\ are\ equal):$ 2) $Q_a'(a) = f'(a)$

$(at\ a, the\ "curvatures"\ are\ equal):$ 3) $Q_a''(a) = f''(a)$

Here is another idea we will use:

The general quadratic function may be expressed as

$$f(x) = k_0 + k_1(x-a) + k_2(x-a)^2$$

We use $(x - a)$ instead of x since

1. $(x - a)$ already appears in the local linearization formula.
2. Using $(x - a)$ rather than x facilitates the ultimate use of the formula and simplifies the following derivation.s

So we will derive a formula for $Q_a(x)$ in the form

$$Q_a(x) = k_0 + k_1(x-a) + k_2(x-a)^2 \qquad (*)$$

Letting $x = a$, we obtain $f(a) = Q_a(a) = k_0 + k_1(a-a) + k_2(a-a)^2 = k_0$

Also $Q_a'(x) = k_1 + 2k_2(x-a)$ So $f'(a) = Q_a'(a) = k_1 + 2k_2(a-a) = k_1$

Finally, $Q_a''(x) = 2k_2$, so $Q_a''(a) = 2k_2$, and $\frac{1}{2}Q_a''(a) = \frac{1}{2}f''(a) = k_2$

Summarizing: $k_0 = f(a),\quad k_1 = f'(a),\ and\ \ k_2 = \frac{1}{2}f''(a)$

Then substituting the *k's* in equation (*), we obtain:

$$Q_a(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

This is called the (Quadratic) ***Taylor Polynomial*** for the function $f(x)$ at $x = a$.

Note that the first two terms of the quadratic approximation together constitute the linear approximation. Thus, the quadratic approximation may be thought of as an "adjustment" to the linear approximation.

Example: a) Find the quadratic approximation of the function $f(x) = \sin x$ at $x = \frac{\pi}{2}$.

b) Graph in several windows to see how the function and its quadratic approximation are related.

c) Find sin(1.3) and sin(1.5) using the Quadratic Approximation, and compare these to the calculator-generated values of sin(1.3) and sin(1.5).

$f'(x) = \cos x$, and $f''(x) = -\sin x$

So $k_0 = f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1,$

$k_1 = f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0,$ and

$k_2 = \frac{1}{2}f''\left(\frac{\pi}{2}\right) = \frac{1}{2}(-\sin\frac{\pi}{2}) = -\frac{1}{2}$

Putting this all together, we obtain:

$$Q_a(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

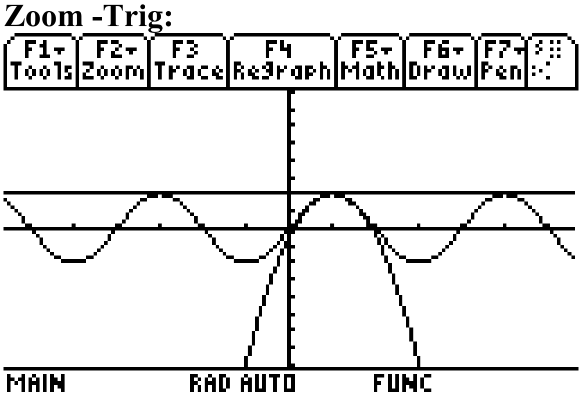
$$Q_{\frac{\pi}{2}}(x) = 1 \quad + \quad 0(x-\frac{\pi}{2}) \quad + \frac{1}{2}(-1)(x-\frac{\pi}{2})^2$$

$$= 1 - \frac{x^2}{2} + \frac{\pi x}{2} - \frac{\pi^2}{8}$$

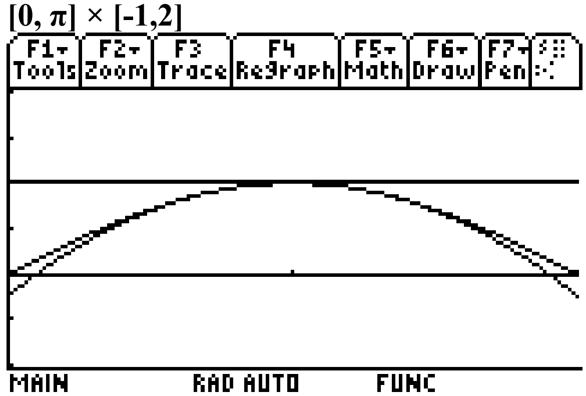
The linear approximation is, of course, just $L_{\frac{\pi}{2}}(x) = 1 + 0(x-\frac{\pi}{2}) = 1$

Showing these graphs on the same axes:

Zoom -Trig:



[0, π] × [-1,2]



$Q_{\frac{\pi}{2}}(1.3) = 1 - \frac{1.3^2}{2} + \frac{1.5708(1.3)}{2} - \frac{3.1416^2}{8} \approx .963335$ Using the calculator: $\sin(1.3) \approx .963558$

$Q_{\frac{\pi}{2}}(1.5) = 1 - \frac{1.5^2}{2} + \frac{1.5708(1.5)}{2} - \frac{3.1416^2}{8} \approx .997494$ Using the calculator: $\sin(1.5) \approx .997495$

The process of finding linear and quadratic approximations is a routine (although somewhat complicated) process. It therefore lends itself quite nicely to a script.

You can enter or download the "**QUADAPPR**" script below. It contains the necessary commands to find the linear and quadratic approximations of any function that you enter in the second line before the "→**f(x)**", at the value of *a* you enter in the fourth line before the "→**a**".

```
C:NewProb
C:sin(x)→f(x)
C:f(x)→y1(x)
C:π/2→a
C:f(a)+(d(f(x),x)|x=a)*(x-a)→l(x)
C:l(x)→y2(x)
C:l(x)+(d(f(x),x,2)|x=a)*(x-a)^2/2→q(x)
C:q(x)→y3(x)
```

Remember that to execute the script for any function you choose:

1) Press **[APPS] - 8** (Text Editor), **Choice 2** (Open), **[ENTER]**
2) **Choice 2**
3) choose the folder in which the script is stored (probably **Main**)
4) choose the variable (script) **quadappr**
5) **[ENTER] [ENTER]**

You now see the script.

6) Press **[F3]** (View) and Choose **option 1:** (Script View)

This gives you a split screen with the script on top, and the home screen on the bottom. The upper window has a heavy rectangle around it, signifying that it is the active window on the screen. Press **[2nd]-[APPS]** to change active windows.

7) Move the cursor to the second line, and change the function shown (before the "→") to the one for which you wish to generate the linear and quadratic approximations.
8) Move the cursor to the fourth line and change the *a*-value (before the '→') to the one at which you want the approximations to be generated.
9) Move the cursor to the ***beginning*** of the top line (before **NewProb**) and press **[F4]** to execute each line in turn.

The result of each command will appear in the bottom screen (the **HOME** screen). When you see **q(x) →y3(x)** in the home screen, the script has run its course. Now press **[F3] (View)**, **choice 2 (Clear Split).**

10) Press **[2nd]-[QUIT]** to return to the **HOME** screen.

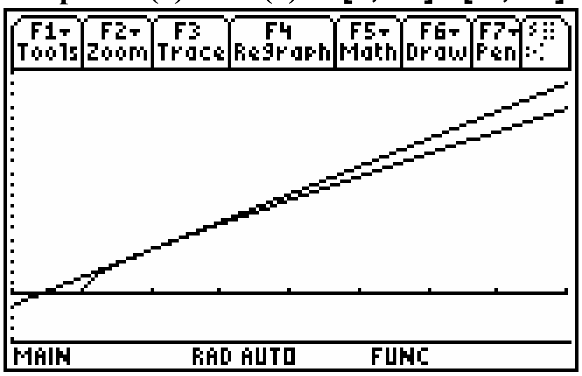
You may now display the linear approximation by typing **l(x)** in the command line. Enter **q(x)** to display the quadratic approximation.

You may also use the calculator to graph the function and its linear and quadratic approximations, because the script stores $f(x)$ in $y1(x)$, $l(x)$ in $y2(x)$, and $q(x)$ in $y2(x)$. Graphing the linear and especially the quadratic approximation is very slow, because first and second derivatives must be calculated for each point.

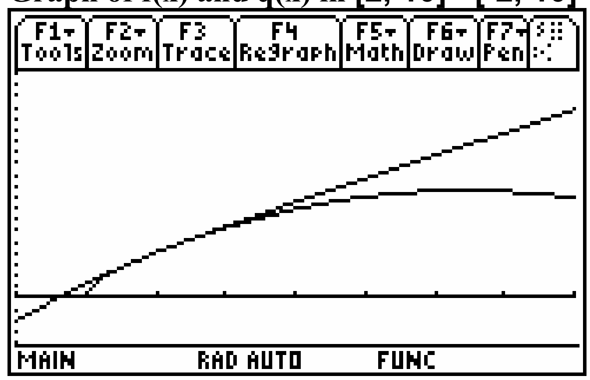
As a trial run of the script,

a) Execute it for the function $f(x)=\sqrt{x^2-3x}$ with $a=4$.
b) Then graph, and use the linear and quadratic approximations to estimate $f(5)$ and $f(4.1)$.
c) Compare these values to the actual values obtained by evaluating $f(5)$ and $f(4.1)$ directly on the **HOME** screen.

Linear approximation
Graph of f(x) and l(x) in [2, 10] ×[-2, 10]



Quadratic approximation
Graph of f(x) and q(x) in [2, 10] ×[-2, 10]



You should have $L_4(x):\ l(x)=\frac{5x}{4}-3$ and $Q_4(x):\ q(x)=\frac{-9x^2}{64}+\frac{19x}{8}-\frac{21}{4}$ (**)

Using command line you can calculate:

$f(5)\approx 3.10938$	$f(4.1)\approx 2.122368$
$l(5)=3.25$	$l(4.1)=2.125$
$q(5)\approx 3.10938$	$q(4.1)\approx 2.12359$

Higher Order Taylor Polynomials

The cubic Taylor Polynomial contains a fourth term involving $f'''(x)$ *and* $(x-a)^3$. (see problem 4) "Higher order" Taylor Polynomials contain similar additional terms.

The n^{th} degree Taylor Polynomial

$$P_a(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)$$

There is a TI-89 command `taylor` that returns the Taylor Polynomial for any function.:

```
taylor(function, variable, order[, value])
```

See the TI-89 manual for more information and limitations. You may download the complete TI-89® manual from Texas Instruments™. Follow the link on the Math 161 page of the website www.mathbykoehler.com.

To calculate the second-order Taylor Polynomial of $f(x)=\sqrt{x^2-3x}$ with $a=4$, enter the command

`taylor(√(x^2-3),x,2,4)` The calculator returns: $\frac{-9(x-4)^2}{64}+\frac{5(x-4)}{4}+2.$

Expand this ***using the expand command:*** `expand(taylor(√(x^2-3),x,2,4))`

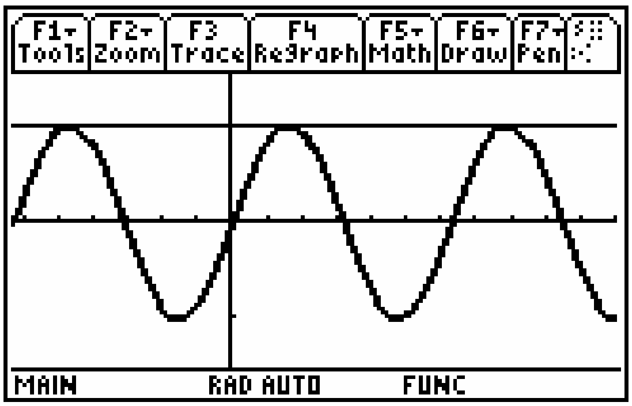
This yields the result $\frac{-9x^2}{64}+\frac{19x}{8}-\frac{21}{4}$, which we obtained as $Q_4(x)$ above. (**)

The “complete” Taylor Polynomial (actually called the *TAYLOR SERIES*) for a function contains an infinite number of terms and higher order derivatives following the pattern established above.

The Taylor series is commonly used to generate tables of trigonometric functions (and others). To demonstrate this, we show graphs of the Sine curve and the associated Taylor polynomials of degrees 1, 2, 4, and 6 that approximate the sin curve at $x = \frac{\pi}{2}$. The graphs are shown in the window $[-2\pi, \frac{7\pi}{2}] \times [-1.5, 1.5]$. Notice that as the order of the Taylor Polynomial increases, its graph more closely resembles the graph of $y = \sin x$ for longer intervals.

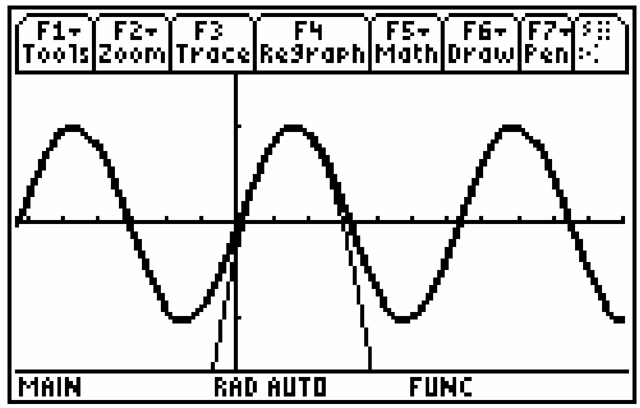
$y = \sin x$ and first order Taylor Polynomial

$$y = 1$$



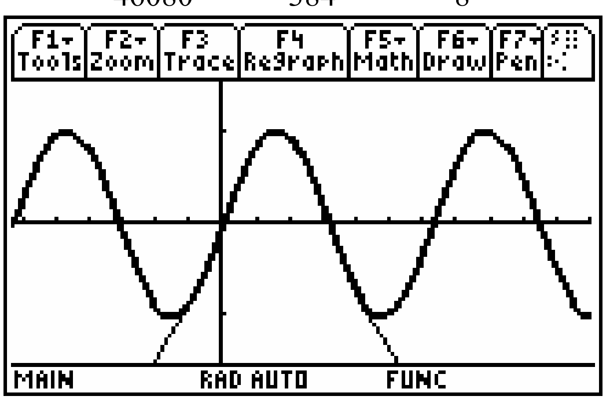
$y = \sin x$ and second order Taylor Polynomial

$$y = -\frac{(2x-\pi)^2}{8} + 1$$



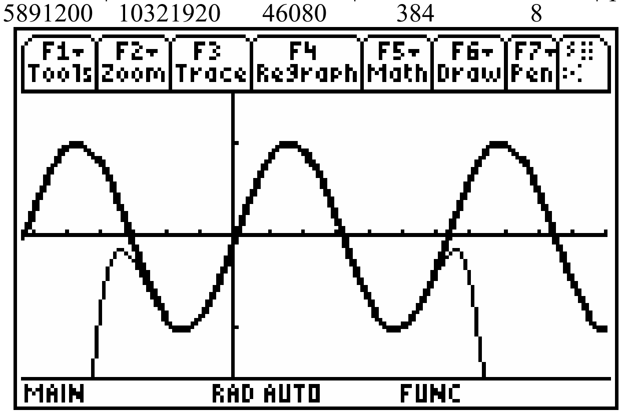
$y = \sin x$ and 6[th] order Taylor polynomial

$$y = \frac{-(2x-\pi)^6}{46080} + \frac{(2x-\pi)^4}{384} - \frac{(2x-\pi)^2}{8} + 1$$



$y = \sin x$ and 11[th] order Taylor polynomial

$$y = \frac{-(2x-\pi)^{10}}{3715891200} + \frac{(2x-\pi)^8}{10321920} - \frac{(2x-\pi)^6}{46080} + \frac{(2x-\pi)^4}{384} - \frac{(2x-\pi)^2}{8} + 1$$



The following table shows the four-place values generated by $y = \sin x$ and by the sixth degree Taylor Polynomial (with $a = \frac{\pi}{2}$) from $x = 0$ to $x = 3.5$.

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$\sin x$	.0000	.0998	.1987	.2955	.3894	.4794	.5646	.6442	.7174	.7833	.8415
Taylor	.0008	.0993	.1984	.2954	.3893	.4794	.5646	.6442	.7174	.7833	.8415

(The Taylor and Sin function values continue to be identical (to four places) until $x = 2.7$)

x	2.5	2.6	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
$\sin x$	.9585	.5155	.4274	.3350	.2392	.1411	.0416	-.0584	-.1577	-.2555	-.3509
Taylor	.9585	.5155	.4273	.3349	.2390	.1407	.0409	-.0596	-.1597	-.2585	-.3553

Using a higher order Taylor Polynomial will generate values that are even closer and for a longer interval. As an example, we suggest that you generate a similar table for $y = \cos x$ from $x = 0$ to $x = \pi$ by tenths, and its 12-degree Taylor Polynomial.

Lab 12 assignment **NAME(PRINT Clearly!)______________________________ LAB _____**

Use the `taylor` ***command to find each approximation.***

1. Consider the function $y = \sqrt{x}$

a) Find the linear and quadratic approximations to the function with $a = 4$.

a) $l_4(x) =$ _______________

$q_4(x)=$_________________________

b) Graph the function and its linear approximation (only) in the window: [-5, 15] × [-1, 5]

c) Graph the function and its quadratic approx. (only) in the window: [-5, 15] × [-1, 5]

d) Enter *l(5)* on the command line to find the linear approximation of $\sqrt{5}$ to four places. d)_____________

e) Enter $q(5)$ on the command line to find the quadratic approximation of $\sqrt{5}$ to four places

e)_____________

f) What value do you get by entering $\sqrt{5}$ at the command line? f)_____________

2. Repeat steps a) – f) from problem 1 for the function $f(x) = 2\cos x$, with approximations at $a = \pi$

a) $l(x) =$ _____________

$q(x)=$______________________

b) Use the window: [0, 6] × [-3, 3]

c) Use the window: [0, 6] × [-3, 3]

d) Enter $l(3)$ on the command line to find the linear approximation for 2cos(3) to 2 dec. pl.

d)_____________

e) Enter $q(3)$ on the command line to find the quadratic approximation for 2cos(3) to 2 dec. pl.

e)_____________

f) What is the value you get by simply entering $2\cos(3)$ on the command line (to 2 dec. pl.)?

f)_____________

3. Given the equation of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=36$

a) Solve the equation for *y* in order to determine the *two* functions that must be graphed in order to show the graph of the entire ellipse.

a)________________________________

____

b) Find the equation of the parabola (quadratic approximation) that best approximates the graph of the ellipse at the point (0, 18).
HINT: use the "positive radical" function from part a) as $f(x)$

b)________________________________

c) Show the graph of the ellipse and its quadratic approximation from part b) in the window [-15.8, 15.8] × [0, 20]

4. Graph $y=\cos x$ and its 14[th] order Taylor Polynomial with $a = 0$ on the same axes in the window $[-2\pi,\frac{7\pi}{2}]\times[-1.5,1.5]$.