Math 163

Lab 10 **_Derivatives and differentiability_**

A ***function is continuous*** if and only if it has no breaks, i.e. if and only if it is possible to move your pencil along the graph from any point on the graph to any other point on the graph without having to lift your pencil from the paper. Formally, mathematicians state:

$$f(x) \text{ is continuous at } x = a \text{ if and only if } \lim_{x \to a} f(x) = f(a).$$

Of course, for this latter equation to be true requires that $\lim_{x \to a} f(x)$ and $f(a)$ both exist.

Visually, this means that as *x* gets close to *a*, the graph of the function $f(x)$ gets close to $f(a)$, so there are no gaps in the graph.

Consider the piece-wise function $f(x) = \begin{cases} 3, & x < 2 \\ -1, & x \geq 2 \end{cases}$. We can graph this using the **when** operator.

SYNTAX of the **when** operator:
when(_true-false condition_, _expression used when condition is true_, _expression when condition is false_)

You may access the **when** operator from the catalog, or by typing it directly. The following example should aid in your understanding of the **when** operator.

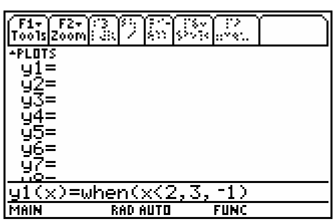
To graph the function $f(x) = \begin{cases} 3 & ,x < 2 \\ -1 & ,x \geq 2 \end{cases}$, press **[♦]-[Y=]** to go to the **y=** screen and for **y1=** enter

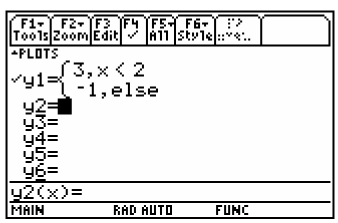
true-false condition

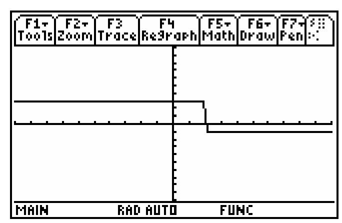
when(x<2,3,-1)

expression used when condition is true, ***expression when condition is false***

This means that when the *x* value is less than 2, the function is equal to 3, and when *x* is not less than 2 (meaning *x* is greater than or equal to 2), the function is equal to -1. Try it.

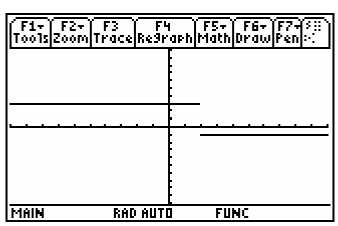






Notice what happens when you enter it: Graph in the standard window

Return to the **[Y=]** screen, Press **[2nd] - [F6] :STYLE**, then choose **2:Dot**. Then press **[♦]** - **[WINDOW]**, set **x-res = 1**, and re-Graph:



This is a fairly accurate picture. If you were drawing the graph by hand, you would put an open circle at the right end of the $y = 3$ portion of the graph, and a solid dot at the left end of the $y = -1$ portion. Press **trace** and enter several values close to 2 to test the function. You will see that for values of 2 or larger, the function always returns -1. When you enter values less than 2, the function always returns 3. Obviously, the graph is not continuous (it has a "jump" discontinuity) at $x = 2$.

Press **[♦] - [Y=]** and enter the command line for this function for **y1**: $\begin{cases} x-1 & ,x \le 2 \\ (x-2)^2 & ,x > 2 \end{cases}$ (Use <= for $\le$)

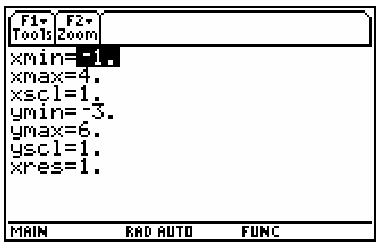
Write the **y1(x)=** command line as it appears on the bottom line of the screen:

____

Write the expression as it appears in the upper portion of the **y=** screen.

$y1 = \{$.

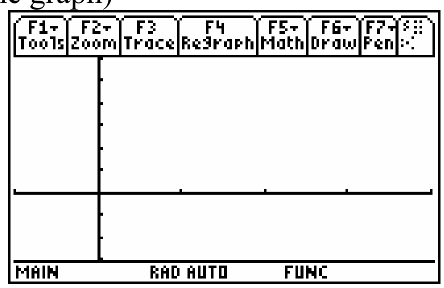
Set your window coordinates as shown below.
Be sure **xres** is set properly.



Then return to the **Y=** screen and highlight **y1=** at the top of the screen. Change the style to **Dot** by pressing **[2nd]- [F6]: Style, choice 2: Dot**

→

Your graph: (***Include open and solid dots*** to indicate which endpoints are included or excluded from the graph)



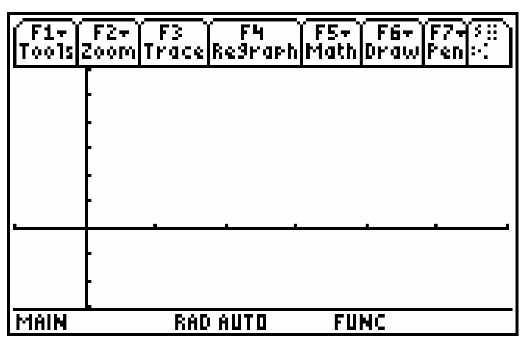
Now clear **y1**, and enter the following as the new **y1(x): when(x < 1, -2,when(x<3,x-2,4-x))**. This "nesting" of **when**s allows for piecewise functions with more than 2 choices.

Write the function as it appears at the top of the screen:

Pay attention to where the **ELSE**s appear

$y1 = \{$.

Graph the function in the
WINDOW [-1,6]×[-3,6].
Include open and solid dots at appropriate endpoints.
Style: DOT, xres = 1



What is the derivative (slope) for each section of the graph?

For $x < 1$, $\frac{dy}{dx} =$ __________ For $1 < x < 3$, $\frac{dy}{dx} =$ __________ For $x > 3$, $\frac{dy}{dx} =$ __________

What about the derivative at the dividing points ($x = 1$ and $x = 3$)? In dealing with continuity at $x = a$, we require the left and right hand limits of the function as x approaches a to be defined and equal. For differentiability at $x = a$, we require that:

Differentiability criteria:

1. The function be continuous at $x = a$, and
2. The left and right hand derivatives be equal at $x = a$.

(There is one other criterion, which will be examined later in this lab.)

The above function is discontinuous at $x = 1$, so it is not differentiable there. It is continuous at $x = 3$, but the left hand and right hand derivatives are not equal (There is a cusp.). Therefore it is not differentiable there. It is differentiable (smooth and not broken) everywhere other than these two places.

A reminder about THE SYMBOLIC DERIVATIVE FUNCTION

The TI-89 has a command to find the symbolic derivative of any function: *d*(. The *d* is italicized. Its syntax is:

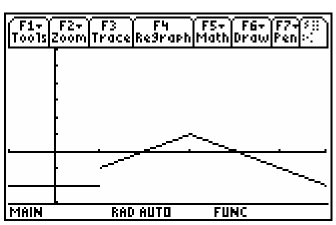
The symbolic derivative SYNTAX: *d*(function, variable)

From the HOME screen the *d*(operator is located in the F3:Calc menu , choice 1. It may also be accessed by way of **[2nd] - 8**, or from the **[CATALOG]**, the first of the "**d**" choices. **Variable** stands for the variable (usually ***x***) you wish to differentiate with respect to.

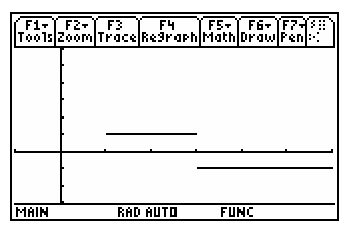
We will use the symbolic derivative to graph the piecewise function, and its derivative

Leave **y1(x)** unchanged. Go to the **Y=** screen and for **y2**, enter: ***d*(y1(x),x)**. Change style to **dot**. Press **F4** to turn off **y2(x)**.

Graph **y1(x)**:



Turn off **y1(x)**, turn on **y2(x)**. Graph



For $x < 1$, the graph of the derivative of the function consists of the line $y = 0$, because the derivative of a constant function is zero. This part actually coincides with the x-axis. Therefore it does not appear on the calculator graph, but trace will work there. Style **THICK** will show this part of the graph, but the pieces will then connect. For $1 < x < 3$, the graph of the derivative consists of the line $y = 1$ because the derivative of $y = x - 1$ is one. For $x > 3$ the graph of the derivative consists of the line $y = -1$ because the derivative of $y = 4 - x$ is -1.

Trace back and forth on **y2** to see the values of the derivative.
While in trace mode determine following derivatives (remember that you can enter an x-value while in TRACE mode to check any value in the window that does not show up exactly when using the cursor keys.)

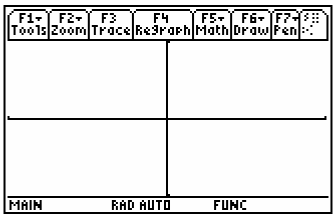
If $x < 1$, then $f'(x) =$ ______ $f'(1)=$______________

If $1 < x < 3$, then $f'(x) =$ ______ $f'(3) =$ ______________ If $x > 3$, then $f'(x) =$ ___________

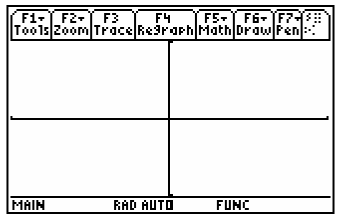
Notice that the derivative does not exist where the graph of the function

1) Has a cusp, or
2) Is discontinuous

Also, the derivative does not exist if a graph has a vertical tangent at a point. Example: $y = \sqrt[3]{x}$ at $x = 0$. Graph $y = \sqrt[3]{x}$ in the window [-1, 1] × [-1, 1] to see this. (It is tangent to the y-axis at the origin.) Also graph the derivative in the same window, and use trace to see the derivative at $x = 0$.



$y = \sqrt[3]{x}$
in [-1, 1]×[-1,1]



derivative of $y = \sqrt[3]{x}$
in [-1, 1]×[-1,1]

$\frac{d}{dx}\left(\sqrt[3]{x}\right) =$ ____________

$f'(0) =$ ____________

In Homework problem #1, you will examine another function having this behavior.

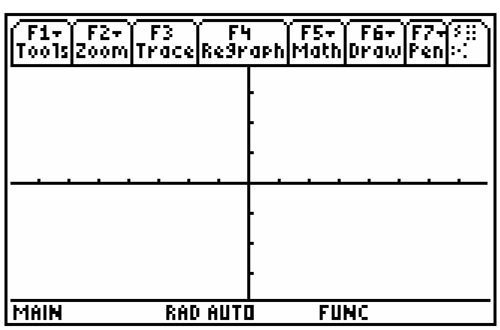
Clear **y1** and **y2**, and return to the **HOME** screen. Enter the command **–3 →c** to store the value –3 in the variable *c*.

Now enter the **Y=** screen, and set $y1 = f(x) = \begin{cases} x+2 & x < -1 \\ x^2 + c & x \geq -1 \end{cases}$. Write the function that you must enter at the bottom of the screen to create this graph.

y1(x) = ____

Write the function y1 as it appears at the top of the screen: $y1 = \{$.

Return to the **Y=** screen, and use the cursor keys to highlight **y1**. Then press **[2nd] – [F6], choice 2** to set **style** to **dot**. Finally, press **[F2] : choice 4: zoom-decimal**. Draw the graph here:



For what value of *x* is the graph discontinuous, and therefore also not differentiable?

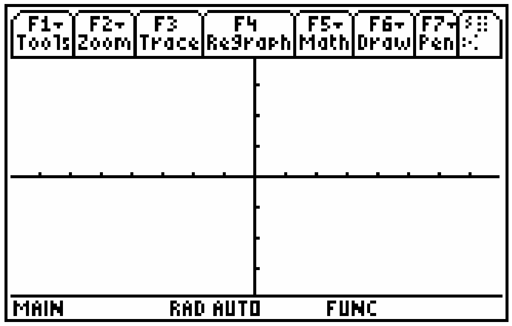
Graph is discontinuous at $x =$ ___

Now you will be investigating the value of c that will make $f(x)$ continuous at this x-value.

For precisely what value of c will the function be continuous? You will have to experiment by returning to the home screen and storing different values in the variable c, and then re-graphing the function to see what value of c causes the 2 parts "join up," i.e., be continuous.

$c =$ ___

Draw the resulting graph (in the **zoomdec** window) when you have found the value of c that causes the 2 parts of the graph to join up.



Is **y1** differentiable at x = - 1? **CIRCLE ONE: <u>YES / NO</u>**

Why or why not? ____

____

____

Making a piecewise function Continuous at the "separation point"

Algebraically, to determine what value of c will make $y1 = f(x) = \begin{cases} x+2 & x < -1 \\ x^2 + c & x \geq -1 \end{cases}$ continuous at, the "separation point" where x = -1, we must have $f(-1) = \lim_{x \to -1^-} f(x)$.

$f(-1) = 1 + c$ and $\lim_{x \to -1^-} f(x) = -1 + 2 = 1$.

In order for the function to be continuous at $x = -1$, $f(-1)$ and the left-hand limit must be equal:

$$\begin{aligned} f(-1) &= \lim_{x \to -1^-} f(x) \\ 1 + c &= 1 \\ c &= 0 \end{aligned}$$

Making a piecewise function Continuous and Differentiable at the "separation point"

Change $y1$ to $y1 = f(x) = \begin{cases} ax+2 & x<-1 \\ x^2+c & x \ge -1 \end{cases}$. Then from the HOME screen, type **1→a** to store 1 in *a*. Re-graph. There should be no change in the graph, since the coefficient of the coefficient *a* was originally 1 and *c* is still the value you found above. If we wish **y1** to be continuous at $x = -1$ (the separation point), then the right and left hand limits of the function at $x = -1$ must be equal. If we also wish *f(x)* to be differentiable there (smooth, continuous, and no cusp) then the left and right hand derivatives at $x = -1$ must also be equal. Therefore, *to ensure differentiability*, we consider the derivatives on each portion of the graph:

For $x < -1$, $f(x) = ax + 2$, so $\boxed{\text{for } x < -1, \quad f'(x) = a}$

For $x \ge -1$, $f(x) = x^2 + c$, so $\boxed{\text{for } x \ge -1, \quad f'(x) = 2x}$

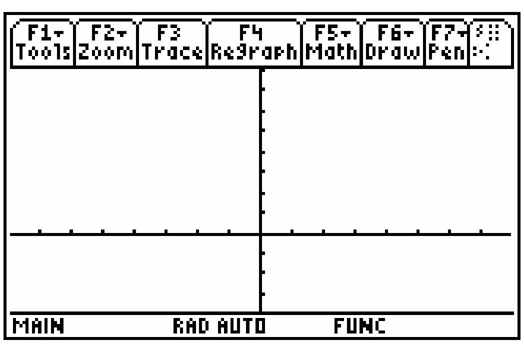
Since we want the graph to be smooth at $x = -1$, i.e. have no cusp at $x = -1$, the derivatives (slopes) of these two functions must be equal at $x = -1$. Therefore, when $x = -1$, we must have $2x = a$. Substituting $x = -1$, we obtain $a = -2$.

Hence we know that $\boxed{f(x) = ax + 2 = -2x + 2, \text{ if } x < -1.}$

Now for continuity: The value of the function itself must approach the same value as *x* approaches -1 from Hence,

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^+} f(x) \\ \lim_{x\to -1^-} (-2x+2) &= \lim_{x\to -1^+} (x^2+c) \\ 4 &= 1+c \\ c &= 3 \end{aligned}$$

Return to the **HOME** screen, and type: **3→c** and **-2→a**. Finally, set **ymax = 8**, and regraph **y1**.



NOTE: In real-life situations, when laying out things like curves on a roadway, or curvature of the tracks on a roller coaster, engineers also require that the second derivatives of the functions for the path of the roadway (as curvature changes from one function rule to another) be equal. This ensures that the transition from a straight or level portion to a curved portion will be smoother.

1. Be sure **MODE** is set to **Complex format: REAL.**

   a) Graph $f(x)=\sqrt[3]{x-2}+3$ and its derivative $f'(x)$, in the window **[-1, 6]** × **[-2, 6], x-res = 1**

   b) Where is $f(x)$ NOT differentiable, and why?

2. Answer *Always*, *Sometimes*, or *Never*:

   a) If a function is continuous at $x = a$, it _____________is differentiable there.

   b) If a function is differentiable at $x = a$, it is _____________continuous there.

3. a) Use the Tangent feature of the Math menu when in graph mode to find an equation of the tangent to $y=\sqrt[3]{x}$ at $x=2$ _______________

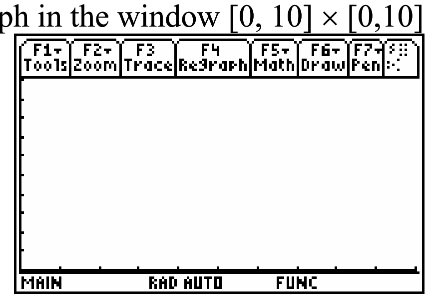
   b) Why does the tangent feature give the result: "No solution found" when you try to find the tangent at $x = 0$? Describe the tangent line at $x = 0$ and give its equation.

State the three ways a function can fail to be differentiable at $x = c$ (both visually and formally).

	Visual	Formal (or algebraic)
a)	______________________________	____
b)	______________________________	____
c)	______________________________	____

4. Determine visually the value of c for which $f(x)=\begin{cases} x^2-2x+c, x<3 \\ x^2+cx-8, x\geq 3 \end{cases}$ is continuous at $x = 3$. Verify this value algebraically by showing that the right-hand and left-hand limits of $f(x)$ at $x = 3$ are the same.

Draw the graph in the window [0, 10] × [0,10]



5. SHOW ALL WORK!! Use the following steps to make a piece-wise function continuous and differentiable at $x = k$:

1. Find the limit of each half of the function as x approaches k. (called the “separation” point)
2. Set these two expressions equal to insure continuity at $x = k$.
   This gives you an equation in a and/or b.
3. Find the left-hand and right-hand derivatives of $f(x)$ at $x = k$.
4. Set these two expressions equal, to insure differentiability at $x = 2$.
   This gives a second equation in a and/or b.
5. Solve the system of 2 equations from steps 2 and 4.
6. Substitute for a and b in the original function $f(x)$, enter it on the **y=** screen, and graph to check that the graph appears reasonable.

Given the function $f(x) = \begin{cases} 2x^2 + 3x + 3, & x < 2 \\ -x^2 + ax + b, & x \geq 2 \end{cases}$.

a) Find the values of a and b for which f is both continuous and differentiable at $x = 2$.

b) Show the resulting graph.

Window: [,] × [,]