Math 163
Lab 7

Graphing Functions and their derivatives; the Symbolic Derivative

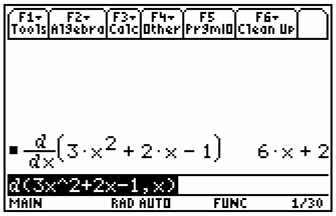
THE SYMBOLIC DERIVATIVE FUNCTION

The TI-89 has a command to find the symbolic derivative of any function: *d*(. The *d* is italicized. Its syntax is:

The symbolic derivative SYNTAX: ***d*(function, variable[,order])**

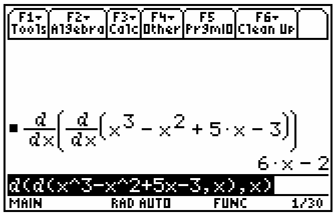
From the **HOME** screen the ***d*(** operator is located in the **[F3]:Calc** menu , choice **1**. It may also be accessed by way of **[2nd]-8**, or from the **[CATALOG]**, the first of the "d" choices. Variable is the variable (usually ***x***) you wish to differentiate with respect to.

For example, to find $\frac{d}{dx}\left(3x^2+2x-1\right)$, you type *d*(3x^2+2x-1,x). The calculator will return 6x+2. Here are some examples:



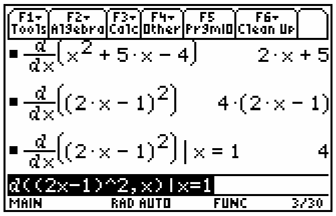
$\frac{d}{dx}(3\cdot x^2+2\cdot x-1)$ $6\cdot x+2$

`d(3x^2+2x-1,x)`



$\frac{d}{dx}\left(\frac{d}{dx}(x^3-x^2+5\cdot x-3)\right)$ $6\cdot x-2$

`d(d(x^3-x^2+5x-3,x),x)`



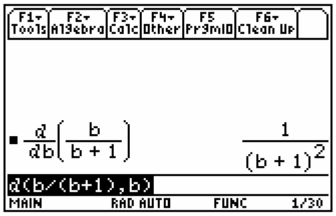
$\frac{d}{dx}(x^2+5\cdot x-4)$ $2\cdot x+5$

$\frac{d}{dx}((2\cdot x-1)^2)$ $4\cdot(2\cdot x-1)$

$\frac{d}{dx}((2\cdot x-1)^2)|x=1$ 4

`d((2x-1)^2,x)|x=1`

You try some others – complicated ones, trig functions, etc. The form may be unexpected, but it does give a correct answer. You may use variables other than *x*.



$\frac{d}{db}\left(\frac{b}{b+1}\right)$ $\frac{1}{(b+1)^2}$

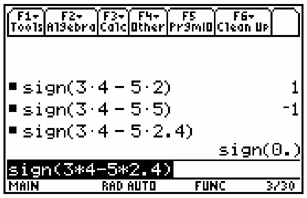
`d(b/(b+1),b)`

THE sign *FUNCTION*

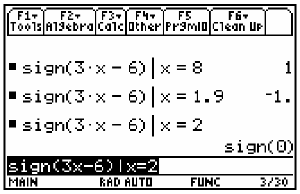
SYNTAX: **sign(expression)**

This function returns +1 if the expression inside parentheses is positive, -1 if the expression is negative, and is undefined if its argument is equal to zero. It just returns **sign (0)** if the expression is equal to zero.

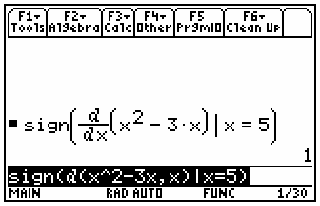
The **sign()** function may be directly typed, found in the [catalog], or in the [MATH] menu (**[2nd] | [5] | choice 1 | choice 8**). As with other functions, it may be combined in larger expressions with other functions.



```
sign(3·4 - 5·2)            1
sign(3·4 - 5·5)           -1
sign(3·4 - 5·2.4)
                     sign(0.)
sign(3*4-5*2.4)
```



```
sign(3·x - 6)|x = 8          1
sign(3·x - 6)|x = 1.9      -1.
sign(3·x - 6)|x = 2
                       sign(0)
sign(3x-6)|x=2
```



$\mathrm{sign}\left(\frac{d}{dx}(x^2-3\cdot x)|x=5\right)$ 1

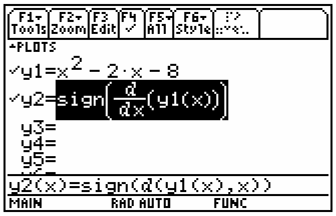
`sign(d(x^2-3x,x)|x=5)`

INVESTIGATING A FUNCTION AND ITS RELATION TO ITS DERIVATIVES

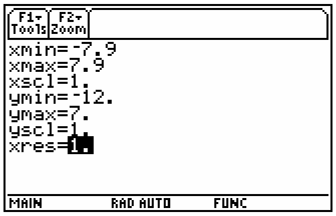
Now we will combine the ***d*** and **sign** functions and investigate the relationship between the behavior of the function and the sign of its derivative.

From your knowledge of the derivative, you will recall that if the derivative of a function is positive on an interval, then the function itself is increasing on the interval. Conversely, if the function is increasing on an interval, then the derivative is positive on the interval.

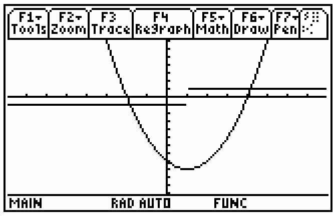
Enter **y1** and **y2** as shown below. Highlight **y2** and set the style of **y2** (only) to **dot**. Set the window coordinates to [-7.9, 7.9] × [-12, 7], with xres = 1. Then graph **y1** and **y2** on the same axes.



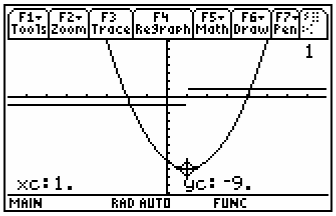
```
PLOTS
✓y1=x^2 - 2·x - 8
✓y2=sign(d/dx(y1(x)))
 y3=
 y4=
 y5=
y2(x)=sign(d(y1(x),x))
```

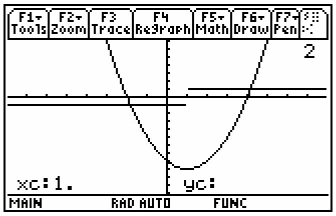


```
xmin=-7.9
xmax=7.9
xscl=1.
ymin=-12.
ymax=7.
yscl=1.
xres=1.
```



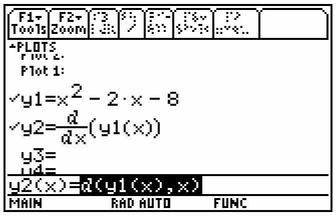
Notice that where y1 is decreasing, the sign of its derivative is negative (because **y2** is -1), and where **y1** is increasing, the sign of its derivative is positive (because **y2** is +1). To investigate the derivative at the vertex of the parabola, where y1 "levels off," use the **TRACE** option on y1 and move the cursor to (1, -9). Recall that if the value of the expression is zero, then the **sign** function returns no value (i.e. is undefined). Press the up or down cursor key to see that the sign of the derivative (**y2**) at the vertex is undefined (blank). This means that the derivative is zero:



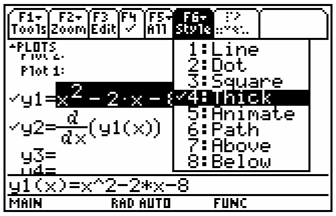


Now we will investigate the actual value of the derivative (rather than just its sign, as above) and its relation to the shape of the function.

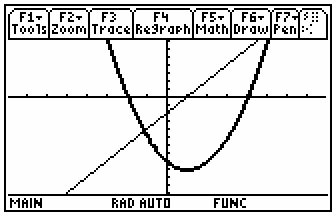
Change the style of **y1** to thick (from the **[Y=]** screen, highlight **y1,** then choose **[F6], choice 4:Thick**). Then change **y2** as shown below. Change the style of **y2** to line.



```
PLOTS
Plot 1:
✓y1=x^2 - 2·x - 8
✓y2=d/dx(y1(x))
 y3=
y2(x)=d(y1(x),x)
```

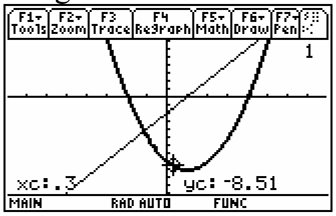


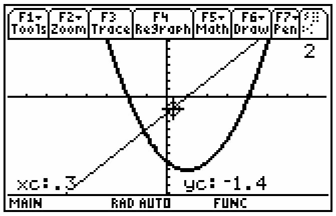
```
PLOTS
Plot 1:
✓y1=x^2 - 2·x - 
✓y2=d/dx(y1(x))
 y3=
y1(x)=x^2-2*x-8
```



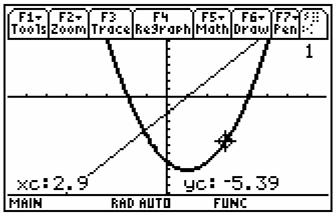
Use TRACE to study the graph carefully. Notice four things:

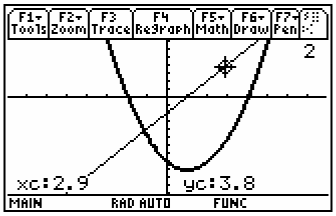
1. When the function (thick graph) is decreasing, the graph of the derivative is BELOW the *y*-axis, so the derivative is negative there.



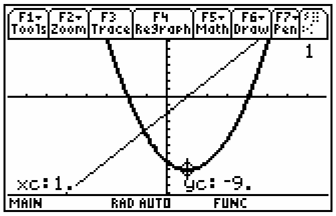


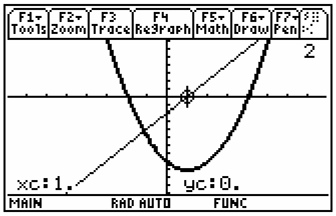
2. When the function (thick graph) is increasing, the graph of the derivative is ABOVE the *y*-axis, so the derivative is positive there.





3. When the function (thick graph) levels off at its vertex, the graph of the derivative has an *x*-intercept, so the value of the derivative is zero there.

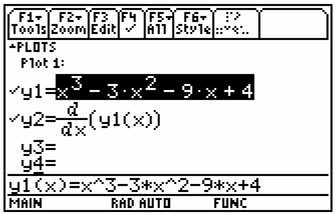




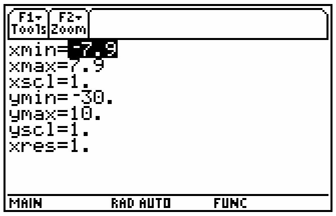
4. As the graph rises or falls more and more steeply, the absolute value of the derivative increases.

Notice that at $x = 1$, the derivative is zero. To the left of $x = 1$, the derivative is negative. To the right of $x = 1$, the derivative is positive. Therefore the function's behavior is decreasing to the left of 1, level at $x = 1$, and increasing for $x > 1$. Hence at $x = 1$ is a RELATIVE MINIMUM.

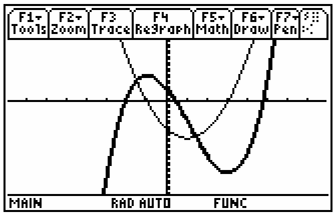
Change **y1** as follows. Then re-set the window, and re-graph.



▪PLOTS
Plot 1:
✓y1=$x^3-3\cdot x^2-9\cdot x+4$
✓y2=$\frac{d}{dx}$(y1(x))
y3=
y4=
y1(x)=x^3-3*x^2-9*x+4

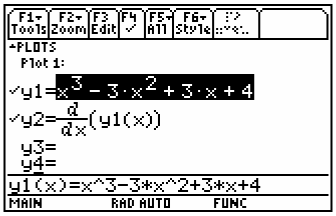


xmin=-7.9
xmax=7.9
xscl=1.
ymin=-30.
ymax=10.
yscl=1.
xres=1.

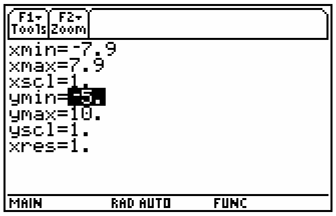


Note that as *x* moves left to right through $x = -1$, the derivative changes sign from positive (above the x-axis), to zero (has an x-intercept), to negative (below the x-axis), and the graph changes from increasing to level to decreasing. Thus at $x = -1$ there is a LOCAL MAXIMUM.

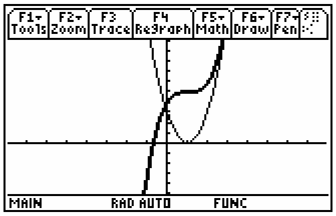
As a further investigation with the derivative, change y1 as follows.



```
*PLOTS
 Plot 1:
✓y1=x^3 - 3·x^2 + 3·x + 4
✓y2=d/dx(y1(x))
 y3=
 y4=
y1(x)=x^3-3*x^2+3*x+4
```

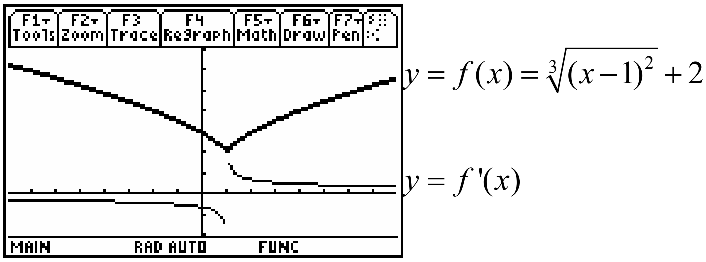


```
xmin=-7.9
xmax=7.9
xscl=1.
ymin=-5.
ymax=10.
yscl=1.
xres=1.
```



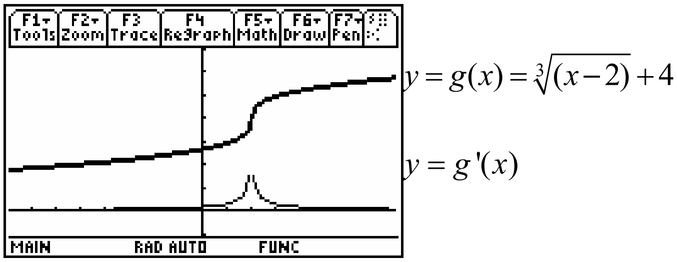
Notice that at $x = 1$, the derivative changes from positive to zero and back to positive. There is no local extreme, because at $x = 1$ is neither a high nor a low point on the graph. At $x = 1$, we say there is a "ledge."

Now consider the graph of $y = \sqrt[3]{(x-1)^2} + 2$ in the window **[-7.9, 7.9]×[-2, 7]**, together its derivative:



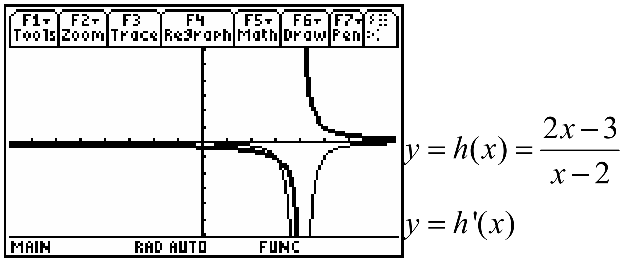
At (1, 2), the graph has a cusp. and by using **TRACE** on the derivative, you will see that the derivative is undefined at $x = 1$. To the left of $x = 1$, the derivative is negative (so $f(x)$ is decreasing) and to the right of $x = 1$, the derivative is positive (so $f(x)$ is increasing). Thus, the point (1, 2) is a relative minimum of the function. *i.e.* the derivative goes "neg → undef → pos" around $x = 1$.

Another situation: Consider $y = g(x) = \sqrt[3]{x-2} + 4$ and its derivative in the window: **[-7.9, 7.9]×[-1, 5]**



At the point (2, 3), $g(x)$ has a vertical tangent. Note that the derivative is undefined there. Since the graph is increasing on both sides of $x = 2$, the point (2, 3) is NOT a relative extreme of g. *i.e.* the derive "goes "pos → undef → pos" around $x = 2$.

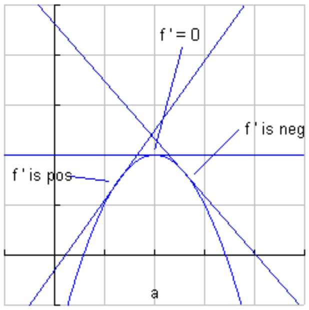
Finally, consider $h(x) = \dfrac{1}{x-4}$ and its derivative in the window **[-7.9, 7.9] × [-10, 10]**



Here there are two reasons that $h(x)$ has no relative extreme at $x = 2$.

1. The function is not defined at $x = 2$
2. The function is decreasing on both sides of $x = 2$. *i.e.* the derivative "goes" neg → undef → neg around $x = 2$

First derivative test for local maximum:

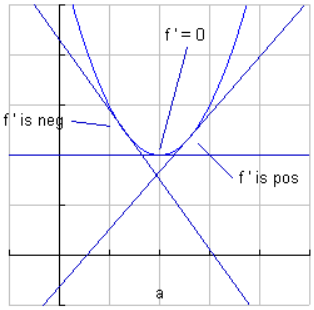


If ,

1. $f'(a) = 0$ or is undefined
2. *f* is increasing ($f'(x)$ is positive) a little to the left of $x = a$, and
3. *f* is decreasing ($f'(x)$ is negative) a little to the right of $x = a$,

then at $x = a$ is local (or relative) maximum of *f*. (Notice that the function goes (/ ¯ \ ) around $x = a$.

First derivative test for local minimum:



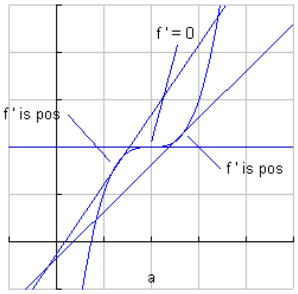
If

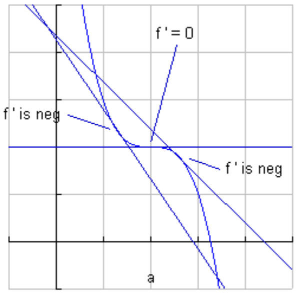
1. $f'(a) = 0$ or is undefined,
2. *f* is decreasing ($f'(x)$ is negative) a little to the left of $x = a$, and
3. *f* is increasing ($f'(x)$ is positive) a little to the right of $x = a$,

then at $x = a$ is local (or relative) minimum of *f*. Notice that the function goes: (\ _ /) around $x = a$.)

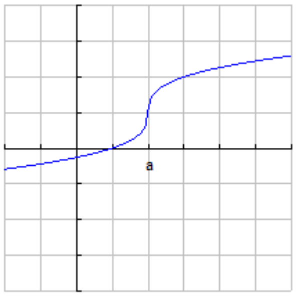
When the first derivative test indicates no relative extremes:

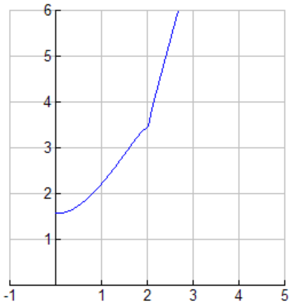
If the derivative goes positive to zero to positive (/ ¯ /) (figure 1) or negative to zero to negative, (\ _ \ ) (figure 2) there is no local extreme.





Here are two situations where the derivative does not change sign at $x = a$, and the derivative is undefined there.





As a result of the previous investigations, we make the conclusion.

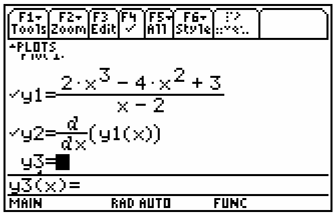
For a local extreme of the function $y = f(x)$ to exist at $\boldsymbol{x = a}$, three criteria must be met:

- ***a*** must be in the domain of ***f***.
- $f'(a) = 0$ or is undefined
- The derivative must have opposite signs on either side of **x = a**.
  - If the derivative goes from positive to zero or undefined to negative around $x = a$, ($f(x)$ has the shape / ¯ \ ), there is a local maximum at $x = a$. The local max is $f(a)$.

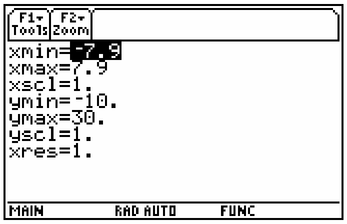
- If the derivative is negative a little to the left of a, positive a little to the right of a, and zero or undefined at $x = a$, ($f(x)$ has the shape: \ _ /) there is a local minimum at $x = a$. the local minimum is $f(a)$.
- If the derivative goes from positive to zero (or undefined) back to positive ($f(x)$ has the shape: /−/) or negative to zero (or undefined) back to negative ($f(x)$ has the shape: \−\ ) there is no local extreme.
- If $f'(x)$ is undefined at $x = a$, but $f(x)$ is defined there, the graph has a cusp or a vertical tangent at that point.

By examining the graph of a function and its derivative, determine the interval(s) where the given function is increasing and where it is decreasing. Give the coordinates of the relative maximum(s) and minimum(s) and equation(s) of any vertical asymptote(s). $y = f(x) = \dfrac{2x^3 - 4x^2 + 3}{x - 2}$

Graph $y1 = f(x)$ and $y2 = d(y1(x), x)$ Change $y1$ to thick style.

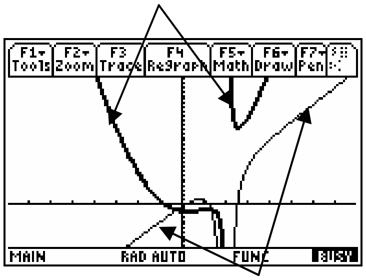


```
F1 Tools | F2 Zoom | F3 Edit | F4 ✓ | F5 All | F6 Style
▲PLOTS
✓y1=(2·x^3 - 4·x^2 + 3)/(x - 2)
✓y2=d/dx(y1(x))
 y3=█
y3(x)=
MAIN        RAD AUTO        FUNC
```



```
F1 Tools | F2 Zoom
xmin=-7.9
xmax=7.9
xscl=1.
ymin=-10.
ymax=30.
yscl=1.
xres=1.
MAIN        RAD AUTO        FUNC
```

$y1 = f(x)$



$y2 = f'(x)$

Study the graph. Notice:

- When $f(x)$ is increasing, the derivative lies above the x-axis (meaning that $f'(x)$ is positive).
- When $f(x)$ is decreasing, the derivative lies below the x-axis (meaning that $f'(x)$ is negative).
- There are three critical values of x, where $f'(x)$ =0.
- At the three critical values points where $f(x)$ levels off, the derivative = 0 (has an x-intercept).
- At $x = 2$, $f'(x)$ is undefined (not continuous), note that the $f(x)$ is also undefined (*i.e.* $f(x)$ is not defined nor differentiable at $x = 2$.), so 2 is NOT a critical value of $f(x)$.
- When the graph of $f(x)$ is increasing and becomes steeper, the derivative increases.
- When the graph of $f(x)$ is decreasing and becomes steeper, the derivative decreases.

FINDING EXTREME VALUES OF A FUNCTION USING THE CALCULATOR.

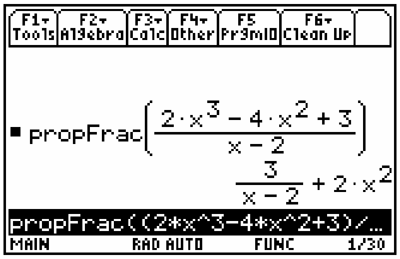
To determine the extreme values of a function when the derivative is equal to zero, you can use the calculator to find the derivative, and then solve the equation "derivative = 0."

1. Enter the function in Y1
2. At the command line, type: solve(***d***("function",x)=0,x) The command
   solve(d((2x*3-4x^2+3)/(x-2),x)=0,x) yields the values .243, 2.543, and 1.214.
   These may be seen to be the x values of the points where the graph in the previoius example levels off.
3. Check where the derivative is undefined (division by zero). Then check any such x-value to see if it is in the domain of the function. If not, it is not a critical value. In the above example, this clearly this gives x = 2, the place where the graph is not continuous. Of course, since 2 is not in the domain of the original function, it is NOT a critical value.
4. You may use the graph of the derivative to determine where the graph of the function is increasing or decreasing. This information, together with critical values, give you the places where the $f(x)$ has local maximums or local minimums.

END BEHAVIOR

Now we investigate how does the function behaves as x approaches positive or negative infinity. This is called “end behavior”. Is there a simpler function that the graph resembles as x gets large? From the graph on the previous page, it appears that if x is larger than 3 or less than –3 it might behave much (but not exactly) like a parabola. We are ignoring, for the moment, the function’s behavior for values close to zero. We are only interested in what happens to the function as x approaches $\pm\infty$, not for finite values of x.

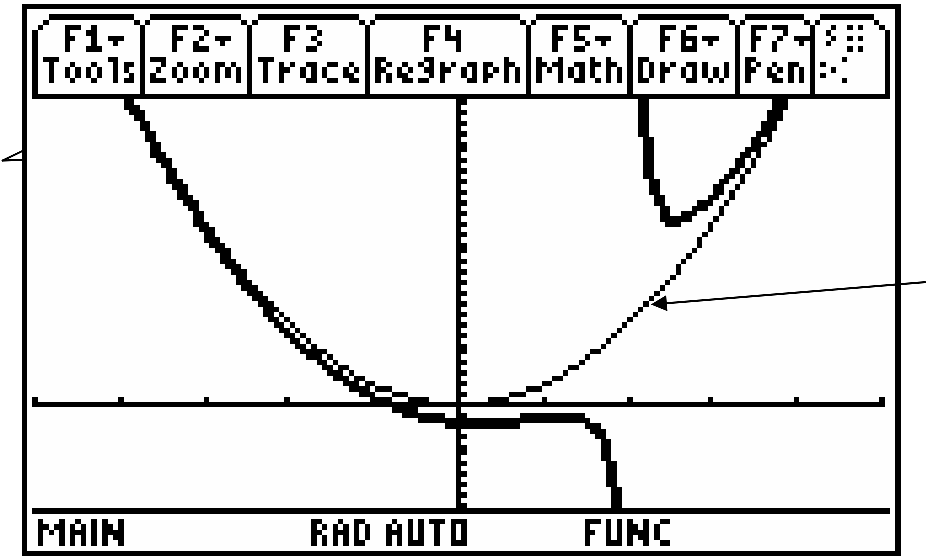
The original function $y = \frac{2x^3 - 4x^2 + 3}{x-2}$ is called “improper” because the degree of the numerator is larger than the degree of the denominator. We change this to a “mixed expression”, use the **propfrac** command in the algebra menu.



■ propFrac$\left(\frac{2\cdot x^3 - 4\cdot x^2 + 3}{x-2}\right)$ $\frac{3}{x-2} + 2\cdot x^2$

```
propFrac((2*x^3-4*x^2+3)/…
MAIN          RAD AUTO      FUNC      1/30
```

Notice that $\lim\limits_{x\to\infty} \frac{3}{x-2}$ is zero, so that $\lim\limits_{x\to\infty}\left(2x^2 + \frac{3}{x-2}\right) = \lim\limits_{x\to\infty} 2x^2$. Hence, as x increases or decreases without bound, the graph of $y = \frac{2x^3 - 4x^2 + 3}{x-2}$ behaves like that of $y = 2x^2$. This can be seen in the graph of the two functions on the same axes below. As increases or decreases without bound, it can be seen that the two graphs get closer and closer. The graph of $y = 2x^2$ is called an “End Behavior Model” of $y = \frac{2x^3 - 4x^2 + 3}{x-2}$.



$y = \frac{2x^3 - 4x^2 + 3}{x-2}$

$y = 2x^2$

Lab 7 Assignment
Answer these on separate paper.

For each problem, enter the given function as **y1** and graph. In all cases use **xmin = -7.9** and **xmax = 7.9.** You will have to adjust **ymin** and **ymax** to see a complete graph. Make your graph a NEAT ONE.

1. Given $y = f(x) = 2x^4 - 3x^3 - 31x^2 + 30x + 56$
   a. Determine **ymin** and **ymax** that show all features of the graph, and sketch. Include window coordinates.

   b. On the same axes used in part *a*, sketch the graph of the derivative of $f(x)$.

   c. Use the graph of the derivative to determine all critical values of the function, and intervals where the function is increasing, where it is decreasing, and where it has local extremes. Use this information to identify the extremes as local maximums or local minimums.

2. Given $y = f(x) = \sqrt[3]{(2x-8)^2} + 3$
   a. Determine **ymin** and **ymax** that show all features of the graph, and sketch. Include window coordinates.

   b. On the same axes used in part *a*, sketch the graph of the derivative of $f(x)$.

   c. Use the graph of the derivative to determine all critical values of the function, and intervals where the function is increasing, where it is decreasing, and where it has local extremes. Use this information to identify the extremes as local maximums or local minimums.

3. Given $y = f(x) = \dfrac{x^4 + x^3 - 6x^2 + 6}{x^2 + x - 6}$
   a. Determine **ymin** and **ymax** that show all features of the graph, and sketch. Include window coordinates.

   b. On the same axes used in part *a*, sketch the graph of the derivative of $f(x)$.

   c. Use the graph of the derivative to determine all critical values of the function, and intervals where the function is increasing, where it is decreasing, and where it has local extremes. Use this information to identify the extremes as local maximums or local minimums.

   d. *Extra Credit:* Find the "end behavior model" for this graph, and sketch it on the same axes as the given function.