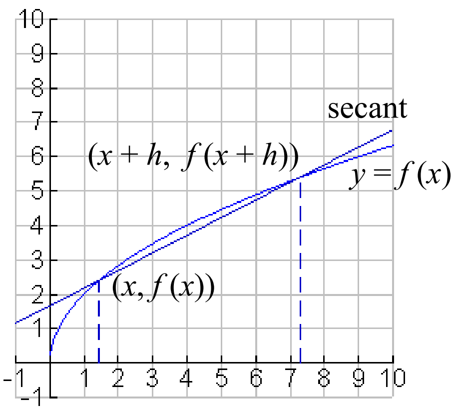
Average Rate of Change and the Derivative. Local Linearity

AVERAGE RATE OF CHANGE OF A FUNCTION

To find the average rate of change of a function (or slope of a secant), from the point $(x, f(x))$, to the point h units to the right: $(x + h, f(x + h))$, use the **avgRC** function:

SYNTAX: **avgRC(function, variable[, h])| x=value**.



For **function**, you may explicitly state the function, or use a function stored in ***y*1(*x*)**, ***y*2(*x*)**, etc., or a function you have defined with the **Define** command.

Variable is the independent variable in the function. .Entering **h** is optional (If not included, the calculator uses $h = .001$.) The "**|**" is shorthand for "when". (It is the 3rd key above the **ON** key.) It causes x temporarily to take on the indicated value.

Let's now consider the function $y = x^3$. To calculate the average rate of change of the function $f(x) = x^3$, with $h = .1$, when $x = 2$, use **AvgRC(x^3, x, .1)|x=2** This average rate of change is, of course, also the slope of the secant to the curve joining the points $(2, f(2))$ and $(2.1, f(2.1))$. The result is 12.61.

NOTE: This is actually the same as the value of $\dfrac{f(x+h)-f(x)}{h}$ when $x = 2$, and $h = 0.1$.

Example 1: Find the average rate of change of $f(x) = x^3 - 2x + 3$ from $x = .5$ to $x = 2.3$

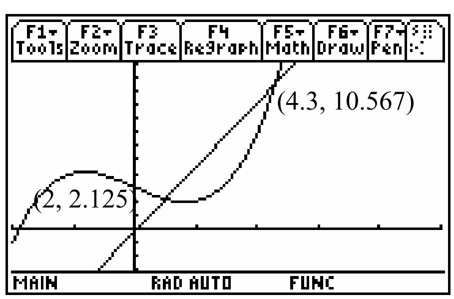
Use the command **AvgRC(x³ – 2x +3,x,1.8)|x=.5** Result: 4.69

Alternatively, you may find the points $(.5, f(.5))$ and $2.3, f(2.3))$ and use the slope formula:

$f(.5) = 2.125$ and $f(2.3) = 10.567$. so the two points are thus (2, 2.125) and (4.3, 10.567)

The slope is then $\dfrac{10.567 - 2.125}{2.3 - .5} \approx \underline{4.69}$

Note that this value, 4.69, is the slope of the secant to the graph of $f(x)$ that passes through the graph at the points (2, 2.125) and (4.3, 10.567). The function and this secant are graphed below:



[-2,5] × [-3, 12]

Example 2: *A table, rather than an algebraic function definition is given*. Consider the data given numerically in the table (rather than algebraically as a function):

x	3	3.5	4	4.5	5	5.5	6
f(x)	18	16	15	12	10	8	7

Find the average rate of change of the function $f(x)$ from $x = 3.5$ to $x = 5.5$

There is no function given for this, so we must simply find the slope using the x and $f(x)$ values in the table:

$$avgRC = m = \frac{8-16}{5.5-3.5} = -4$$

We may use this as an estimate of the slope of the tangent to the graph of $f(x)$ at $x = 4.5$. Other x-values "around" 4.5 could also be used, like 4 and 5, or 3 and 5, etc., to generate an estimate.

Using -4 as the slope, write an approximate equation of the tangent to the graph of $f(x)$ at $x = 4.5$. (at the point (4.5, 12) with slope = - 4). Remember point-slope form: $y - y_1 = m(x - x_1)$

$$y - 12 = -4(x - 4.5)$$
$$y = -4x + 30$$

Use this equation to estimate f (3.8)

$$f(3.8) \approx -4(3.8) + 30 = -15.2 + 30 = 14.8$$

Now we examine the idea that the slope of the tangent line (the derivative) of a function at a particular point is the limit as h approaches 0 of the slopes of the secants drawn from $(x, f(x))$ to $(x, f(x+h))$

- In the y= window, enter y1(x) = 3x^2 – 1
- Enter the program in your calculator:

Instruction	Hints, where to find it
Deriv()	**APPS-7 ; 3:New** ; in the variable box, type Deriv ; press **[ENTER]** twice
Prgm	
ClrIO	
Disp"Enter value of x at which to find deriv:"	
Input e	
2.→h	Be sure to include the decimal point
Lbl L	**Lbl** is in the **Catalog**
avgRC(y1(x),x,h)|x=e→d	**avgRC** is also in the **Catalog**
Disp d	
h/2→h	
Pause	Pause is in the **Catalog** (Causes program to pause. Press **ENTER** to continue)
If h>1.E-10	To get the **E**, Use the **EE** key (the second key above **ON**)
Goto L	**Goto** is in the **Catalog**
EndPrgm	

Notes: **Lbl L** and **Goto L** form a ***"loop:" a set of instructions that will be repeated over and over until something occurs to exit the loop***. When the program encounters the **Goto L** instruction, execution returns to the **Lbl L** line (**Lbl** means LABEL), and execution resumes with the instruction following the **Lbl**. Notice that the **If** condition is true until **h** is **1E-10** (1×10^{-10}) or less, so the **Goto L** returns execution to **Lbl L** until **h** reaches that small a value. Only when h ≤ 1E-10 will the test be false, thus skipping the **Goto** instruction, and getting to the **EndPrgm**.

The ‘**h**’ is the same h as in the definition of the derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The program calculates and then displays the slope of the secant between the 2 points $(x, f(x))$ and $(x, f(x+h))$. It then halves h and repeats the process over again and again until h $\leq$ 1E-10. The value that the display approaches is the derivative of $f(x)$ at the value of x you entered in the program.

Run the program, using $e = 1$. You should observe that the output values approach 6.

FINDING THE DERIVATIVE SYMBOLICALLY USING THE CALCULATOR

Now we will use the calculator to find the slope of the tangent line to the graph of $y = 3x^2 - 1$ at $x = 1$, symbolically, rather than graphically.

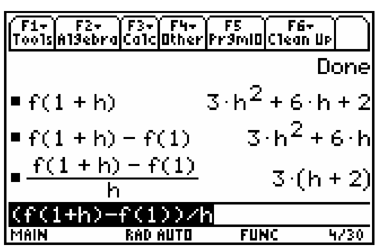
Press **[F5]** to return to the home screen.
Press **[2nd] – [F6], choice 1: Clear a – z** to clear all 1-character variables.

We will now use the calculator to find the derivative symbolically (or analytically) using the definition. Enter the commands as they appear at the left below.

Define f(x)=3x^2-1	
f(1+h)	to begin process to find the derivative at $x = 1$
f(1+h)-f(1)	numerator of the definition of $f'(x)$
(f(1+h)-f(1))/h	

NOTE: the second and third commands above (**f(1+h)** and **f(1+h)-f(1)**) can be omitted, as they simply display steps in working out the difference quotient.

At this point, the screen should look like: (the “**Define f(x)**” line is just off the top of the screen)



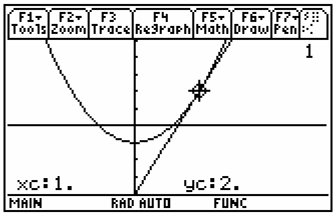
Notice that, as h approaches zero, the value of this final result is 3(0 + 2) = 6. This is the value of the derivative when $x = 1$, and hence the slope of the tangent at $(1, f(1))$ is 6.
Now we write the equation of the tangent at this point. Enter f(1) to evaluate $f(1)$: The result should be 2.
Therefore, the point at which the tangent is to be drawn is (1, 2).
Using the point-slope form of the equation of a straight line: $y - y_1 = m(x - x_1)$,
substituting for m, x_1 and y_1, we obtain: $y - 2 = 6(x - 1)$,
or $\underline{\boldsymbol{y = 6x - 4}}$.

Using the calculator, these last few steps could all be accomplished by the command:
solve(y-f(1)=6(x-1),y)

To verify this result visually, go to the **y=** screen and enter

y1 = 3x^2 – 1
y2 = 6x – 4

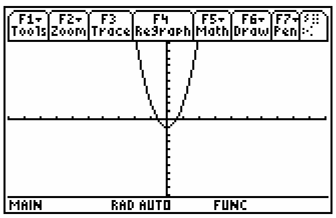
Change the window to **[-2,3]** × **[-4,5]** , graph, and trace to the point (1, 2):



This shows the function (**y1**) and its tangent (**y2**) at the point (1, 2).

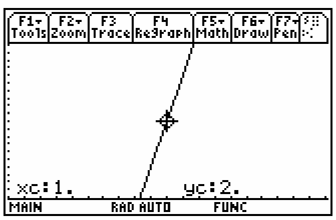
Local Linearity

On the **y=** screen, clear **y2**.
[Zoom]-6 to Graph **y = 3x^2 – 1** in the standard viewing rectangle.



Press **[F3]: Trace** and enter the value 1 to show the point (1,2). Press **[F2] - choice 2** to **Zoom in** on the point (1,2). Repeat this process (**Trace** to the point (1,2) and **Zoom in** on the point) 2 more times.

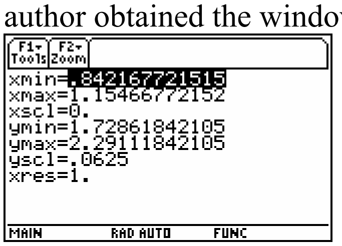
. The graph should eventually look something like:



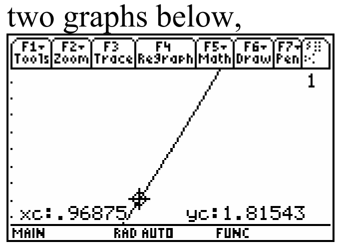
Notice how "straight" this small region of the graph is. As you zoom in repeatedly on any region of a curve, the portion of the graph shown will have less and less "bend". Because of this phenomenon, mathematicians say "Curves are '***locally straight***.'"

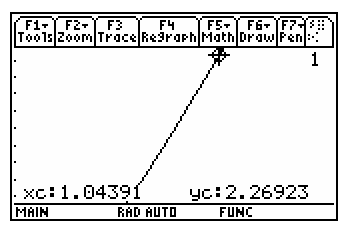
Trace to the left, and record the coordinates of a point near the bottom of the screen. Trace to the right, and record the coordinates of a point near the top of the screen. Calculate the slope of the curve between those two points. It should be about 6. This is the derivative of the function at $x = 1$.

After Zooming in three times, the author obtained the window:



With this window, and using the points as shown in the two graphs below,





We calculate the slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2.26923 - 1.81543}{1.04391 - .96875} \approx 6.037786 \approx 6$.

You zoom in two more times and find the slope in like manner. You should obtain a slope still closer to 6.

result (to 6 dec. pl.) after two more zooms: _______________

THE nDeriv FUNCTION AND THE SYMMETRIC DERIVATIVE

The **nDeriv** function calculates the approximate numeric derivative of a function at any point on the function where the derivative exists.

Syntax: **nDeriv(function, variable[,h])|x= x-value for derivative**

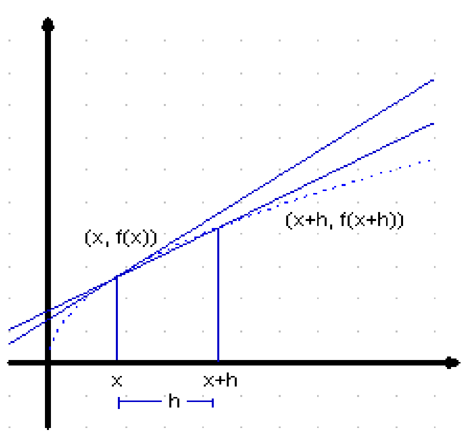
nDeriv is choice A in the **F3: Calc** menu
If h is omitted, it is assumed to be 0.001.

Type **nDeriv(3x^2-1,x)|x=1**. (Read as: The numeric derivative of the function **3x^2-1**, with respect to x, when x is equal to 1.) Since no h value was specified, the calculator uses $h = .001$. The command returns the value 6.

If you calculate the slope by hand using the points $(1, f(1))$ and $(1.001, f(1.001))$, i.e. $x = 1$ and $h = .001$, you will obtain $m = \dfrac{f(1.001) - f(1)}{1.001 - 1} = \dfrac{2.006003 - 2}{.001} = \dfrac{.006003}{.001} = 6.003$. This is close to what you know the derivative to be. However, the **nderiv** function returns exactly 6. The reason is that the calculator is finding the "symmetric" or "two-sided" derivative, $\dfrac{f(1.001) - f(.999)}{1.001 - .999} = \dfrac{2.006003 - 1.994003}{.002} = \dfrac{.012}{.002} = 6.0$. We are used to finding the "one-sided" derivative.

One-sided derivative

When finding the one-sided derivative, we move just one way: from x to x + h to locate a second point to use in finding the slope of the secant



Two sided (Symmetric) derivative

When finding the symmetric, or "2-sided" derivative, we move h to the right and h to the left, (to (x + h) and (x – h), respectively) to locate two points to use in determining the two points to use in finding the slope of a secant. As you can see in the second diagram below, If we use the same value of h this "2-sided" secant is usually a better approximation than the "one-sided" secant
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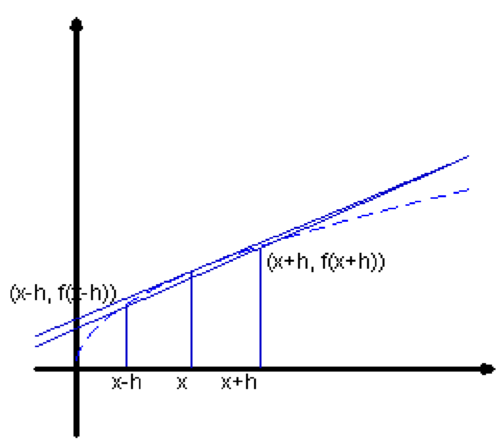


Formally, the symmetric derivative is equivalent to the one-sided derivative, and is defined as follows:

SYMMETRIC DERIVATIVE: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{(x+h) - (x-h)} = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \qquad (1)$$

The symmetric derivative is equivalent to the one-sided derivative. However, with given values of *h,* the two difference quotients will return different, but close values.

Example: Use **nDeriv** to find an estimation of the symmetric derivative of $y = 2x^2 - 1$ at $x = 2$. Use $h = .1$.

nDeriv(2x^2-1, x, .1)|x= 2 Result: 8

Working this out by hand, $(x - h) = 1.9$ and $(x + h) = 2.1$, so:

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{(2(2.1)^2 - 1) - (2(1.9)^2 - 1)}{2(.1)} = 8$$

In essence, the above is simply calculating the slope of the secant line between $(1.9, f(1.9))$ and $(2.1, f(2.1))$. You can see that when the secant line has both of its points on the function moving toward the desired point of tangency (i.e. one point is not anchored at $(2, f(2))$). This causes the slope of the secant to approach the slope of the tangent more quickly than using the one-sided derivative technique. However, the symmetric derivative requires more involved calculations than the one-sided derivative, so we normally use the one-sided derivative in our work by hand. But since the calculator can do both calculations with equal ease, it has been programmed to use the symmetric derivative, which converges to the actual value faster.

Notice that the value the symmetric derivative we got above, 8, is the same value you get when you enter:

On the other hand, if the one-sided derivative is used on $f(x) = 2x^2 - 1$ with $x = 2$ and $h = 0.1$, the result is

$$\frac{f(x+h) - f(x)}{h} = \frac{(2(2.1)^2 - 1) - (2(2)^2 - 1)}{.1} = 8.2\,.$$

This is good evidence that the symmetric derivative usually gives better accuracy than the one-sided derivative, even when *h* is as large as 0.1

 Name(PRINT Clearly!)________________________________ LAB HOUR___

1. Let $f(x) = 2x - x^2$

a) Find the average rate of change of $f(x)$ from $x = 2$ to $x = 2.5$

b) Find the equation of the corresponding secant line through the two points $(2, f(2))$ and $(2.5, f(2.5))$

c) Sketch the graph of f and the secant line from part b).

2. Consider the data:

x	4	4.5	5	5.5	6	6.5	7
f(x)	13	11	10	7	5	3	2

Since you do not know what the algebraic definition of the function is, you will have to use the data in the table above to answer the following questions.

a) use the data in the table above to find the average rate of change of f from $x = 5$ to $x = 7$.

b) use the data in the table above to obtain an *estimate* the instantaneous rate of change of f at $x = 5$ by finding the slope of the secant to the curve between $(4.5, f(4.5))$ and $(5.5, f(5.5))$.

c) Using the answer from b), write an equation of the approximate tangent to $f(x)$ at $x = 5$

d) Use the equation in c) to *estimate* the value of $f(6.5)$

NOTE: This will differ somewhat from the value in the table, as it is a *linear estimate*.

3. Use the process of the example on page 3 at the bottom on this problem.
Be sure Angle Mode is set to **Radian**.
Draw the graph of $y = f(x) = \sin x$ in the window [-4, 4] × [-2, 2], xres=1 **Zoom in** several times on the point(1, sin 1) on the graph until what you see of the graph appears to be straight. (zoom in at least 4 times)

Original window

Final "zoomed window" including the coordinates of the points you used to find the slope.

A) Use the coordinates of two points from the "zoomed window" (refer to the bottom of page 3) to find the slope (to 5 decimal places). This slope is an estimate the derivative of sin x with $x = 1$.

B) How does this slope compare to cos(1)? Why?

4. Given $f(x) = \dfrac{x^2 - 4}{x^2 + 1}$, and the point $(3, f(3))$ on the graph of $f(x)$.

a) Use **nDeriv** to find the derivative of the function at the point

Command line:____ Result__________

b) Use the derivative found in part *a* and the coordinates of the point to find an equation of the tangent line.

c) Draw the graph of the function and the tangent on the same axes.

5. Given the data in the table:

x	1	1.1	1.2	1.3	1.4
$f(x)$	1	0.19	0.27	0.34	0.38

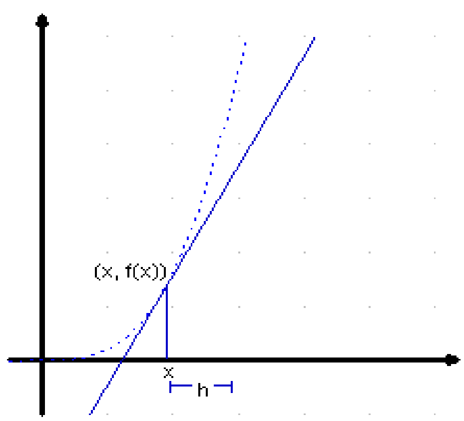
Since you do not know what the algebraic definition of the function is, you will have to use the data in the table above to answer the following questions.

a) use the data in the table above and the ***symmetric derivative*** method with $h = .1$ to estimate $f\,'(1.1)$.

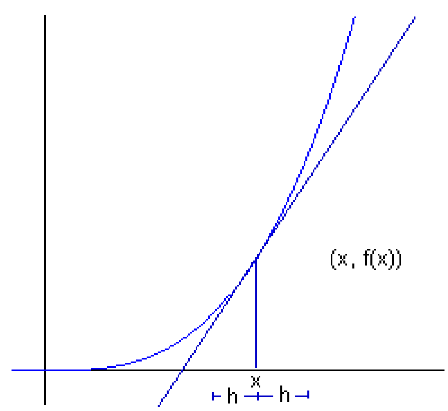
b) Using the value of $f\,'(1.1)$ from part *a),* write an *approximate* equation of the tangent to $f(x)$ at $x = 1.1$.

6. For the (dotted) function and its tangent at $(x, f(x))$ shown below, **complete the diagrams** for the one-sided derivative and the two-sided or symmetric, derivatives:

One-sided Derivative graph



Symmetric Derivative graph



The secant line in the second graph should seem to be closer to the tangent line than does the secant line in the first graph. The limiting positions of each secant will be the same as h approaches zero, but the symmetric secant line "converges" to the tangent line more quickly.