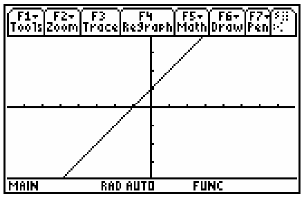
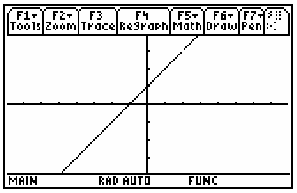
Lab 3
Continuity and Limits

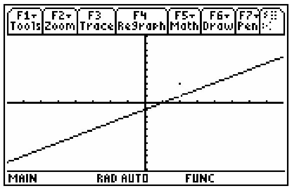
Intuitively, a function is continuous at a point if there is not a break in the graph at that point. For example, graph 1 is continuous at x = 2, but graphs 2 – 6 are not continuous at x = 2.



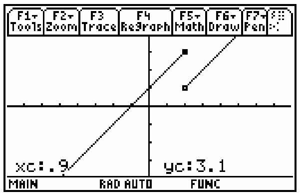
graph 1



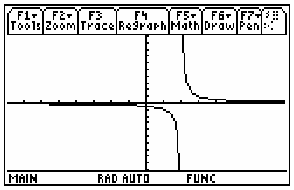
graph 2
notice the gap at (2, 3)



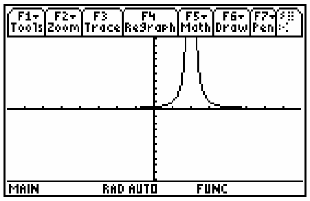
graph 3
(notice the point at (2, 3)
and the gap at (2, 1))



graph 4



graph 5



graph 6

Formally, this requirement is somewhat more difficult to define:

DEFINITION: $f(x)$ is continuous at $x = a$ if and only if

1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

In graph 1, all three are satisfied, because

1. $f(2) = 3$, so the function IS defined at x = 2,

2. & 3. $\lim_{x\to 2} f(x)$ exists and is equal to 3, because as x gets closer and closer to 2, the graph of the function is getting closer and closer to 3, the value of the function at $x = 2$.

In graph 2, $f(2)$ does not exist, although $\lim_{x\to 2} f(x) = 3$. Therefore, $f(2) \neq \lim_{x\to 2} f(x)$. In the homework, you will be asked which conditions are not satisfied in each of the graphs 3 – 6.

We have examined continuity from a graphic viewpoint. In exercise 2, you will investigate it from a numeric viewpoint (a table).

The formal definition of a limit: epsilon and delta.

Now we investigate a bit more formally what is means for a limit to exist. On an intuitive level, the limit of a function $f(x)$ exists at a value, a, of x if and only if you can make the value of $f(x)$ as close as you want to $f(a)$ by choosing x sufficiently close to a. I.e. If you know how close to $f(a)$ you want to be, you must be able to find how close to x must be to a.

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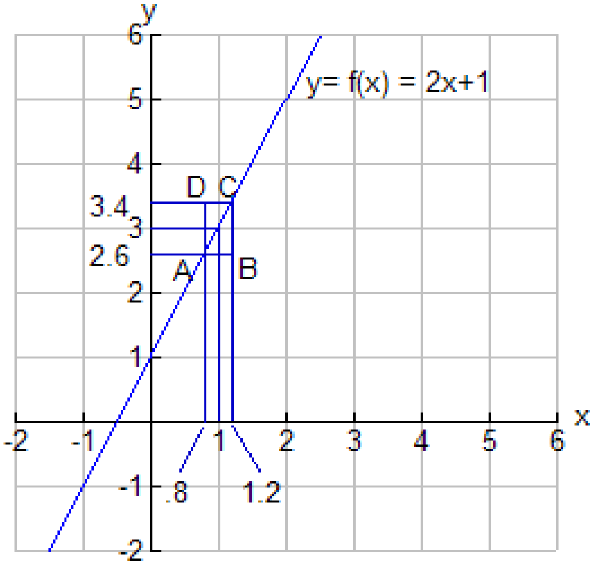
Consider the example $f(x) = 2x + 3$. Clearly, $\lim_{x \to 1} f(x) = 3$. To verify this, we must answer the question: If we want $f(x)$ to be less than some positive distance, say 0.4, away from 3, we must find how close x must be to 1 to guarantee this.

From the graph below it is seen that since we want $f(x)$ to be less than 0.4 away from 3, our goal is to make $f(x)$ between 2.6 and 3.4. To achieve this goal, we make x between 0.8 and 1.2, but not equal to 1. This may be checked by evaluating $f(0.8)$ =2.6 and $f(1.2)$ = 3.4. So if we choose x to be within 0.2 of 1, then $f(x)$ will be within 0.4 of 3.



The vertical distance (0.4) that $f(x)$ is from 3 is called epsilon (ε or $\in$) and the horizontal distance (0.2) that x is from 1 is called delta (δ) (the lower case Greek letter). Examining rectangle ABCD, we observe that the "half-height" is $\varepsilon = 0.4$, so the corresponding "half-width" will be $\delta = 0.2$. Thus, if ε is to be 0.4, we must choose δ to be any value 0.2 or less (but not zero).

The distance between two numbers can be represented by the absolute value of their difference. For example, 3 and 3.4 are .4 apart because |3 – 3.4| = |3.4 – 3| = .4. This may also be stated: "The distance between 3 and 3.4 is .4."

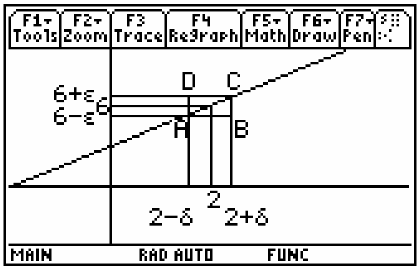
Using absolute value inequalities, the above diagram indicates that: $|f(x) - 3| < .4$ whenever $|x - 1| < .2$

Now refer to the 0.4 as ε and .2 as δ. Then $f(x)$ is within ε of 3 (i.e. $f(x)$ is less than ε from 3) if and only if $|f(x) - 3| < \varepsilon$. Also, x is within δ of 1 (i.e. x is less than δ from 1), but not equal to 1, if and only if $|x - 1| < \delta$ *and* $0 < |x - 1|$. These two inequalities are customarily combined: $0 < |x - 1| < \delta$. Putting all this together, we have the so-called **<u>delta-epsilon definition of a limit</u>**:

$$\boxed{\begin{array}{l} \lim_{x \to a} f(x) = L \text{ if and only if for every } \varepsilon > 0 \text{ there exists a } \delta > 0 \\ \text{such that for all } x, |f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta. \end{array}}$$

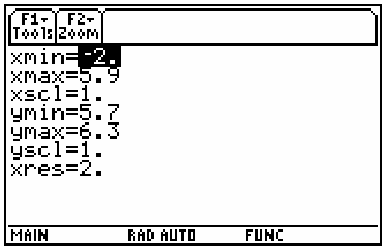
To prove a limit using this definition, we must start with the inequality $|f(x) - L| < \varepsilon$ and transform it to obtain the inequality $|x - a|$ < "*some expression*", usually in terms of ε. This "*expression*" is the δ we are looking for. It establishes the connection between ε and δ. The $0 < |x - a|$ part of the inequality is satisfied if we simply require that x and a are not the same. Let's investigate $\lim_{x \to 2}(\frac{3}{2}x + 3)$. We can easily see that this limit must be 6. Draw the graph of $y = f(x) = \frac{3}{2}x + 3$ in the window [-2, 5.9] × [-4, 10], xscl = 0, yscl=0

<u>Here is the epsilon-delta graph:</u>



graph 7

Let's arbitrarily choose ε = .3 and graphically find the corresponding δ. That means that we are interested in the portion of the graph only between $y = 6 - \varepsilon =$ 5.7 and $y = 6 + \varepsilon$ = 6.3, which is within rectangle ABCD in graph 7. Since we are concerned only with *y* between 5.7 and 6.3, change ymin to 5.7, ymax to 6.3 and re-graph:



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xmin=-2.
xmax=5.9
xscl=1.
ymin=5.7
ymax=6.3
yscl=1.
xres=2.
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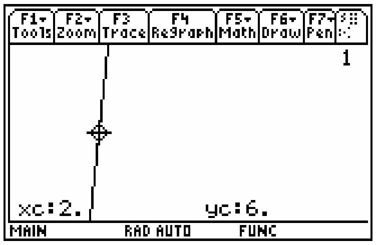
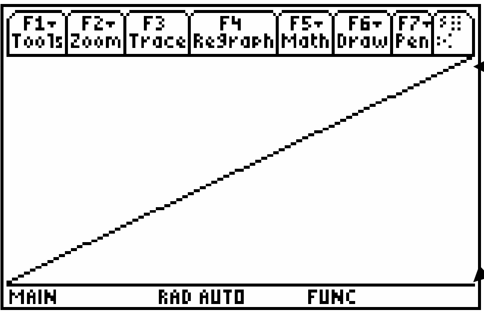


Graph is shown with TRACE "active".

We must find xmin and xmax that will cause the graph to go corner to corner, which will cause rectangle ABCD in graph 7 to fill the entire screen in graph 9. Then xmin = 2 – δ and xmax = 2 + δ .

Now we adjust xmin and xmax so that the graph goes corner-to-corner in the resulting window. This may take you several tries. HINT: TRACE to the top point of the graph in the window above to move to the point whose y-coordinate is exactly 6.3. Use the *x*-value of that point (2.2) as xmax. Repeat the process with the bottom point of graph 8 (the *x*-value there is 1.8) to get xmin. Then reset xmin and xmax and regraph.



D (5.7, 2.2) or
(x - δ , $f(x) + \varepsilon$)

C (6.3, 2.2) or
(x + δ , $f(x) + \varepsilon$)

B (6.3, 1.8) or
(x+δ , $f(x) - \varepsilon$)

A (5.7, 1.8)
or (x - δ , $f(x) - \varepsilon$)

graph 9

Window:
[5.7, 6.3] × [1.8, 2.2]

Graph 9 shows that if $f(x)$ is to be between 5.7 and 6.3, then *x* must be between 1.8 and 2.2. So x - δ = 1.8 and x + δ = 2.2. This gives us δ = 0.2 . Therefore, given an ε = 0.3, we have found the corresponding δ to be δ = 0.2 .

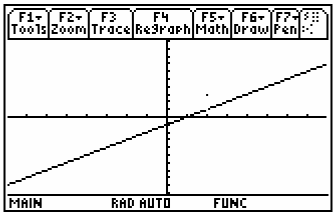
In order to formally prove the limit, this would have to be done in general for every $\varepsilon > 0$, not just ε = .3. You will look at that process in Math 161.

Lab 3 Exercises NAME____________________________

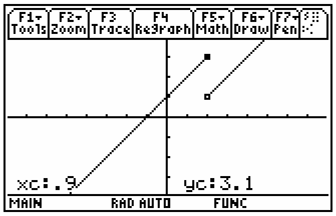
Limits and Continuity

1. For each of the graphs below, state the condition(s) of continuity that are not satisfied at $x = 2$. The conditions for continuity at $x = a$ are listed here for reference.

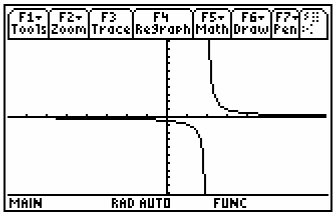
1. $f(a)$ exists 2. $\lim_{x\to a} f(x)$ exists 3. $\lim_{x\to a} f(x) = f(a)$



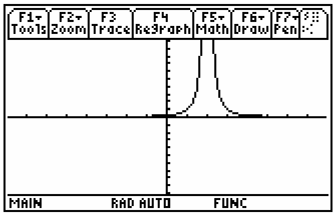
graph A



graph B



graph C



graph D

Condition(s) not satisfied:

graph A ______________________________ graph C ______________________________

graph B ______________________________ graph D ______________________________

2. Given $f(x) = \dfrac{x^3 - 2x^2 + x}{x}$. Enter $f(x)$ as y1. Use TblSet to set independent ° ask. Now examine the table (You may have to clear existing values out of the table first.). Complete this table.

x	-.05	-.01	-.001	0	.001	.01	.05
$f(x)$							

Which condition(s) of continuity are not satisfied for $f(x)$ at $x = 0$?

2. ______________________________

3. Repeat exercise 2 with the function $f(x) = \begin{cases} \dfrac{x^2 - 2x}{x - 2}, & x \neq 2 \\ 5, & x = 2 \end{cases}$

x	1.95	1.99	1.999	2	2.001	2.01	2.05
$f(x)$							

Which condition(s) of continuity are not satisfied for $f(x)$ at $x = 2$?

3.______________________________

4. (Use the back of this page for this problem.)
Follow the steps below to find the δ that corresponds to an ε of 0.6. for $\lim_{x\to 3}(2x-1)$.
   - a) Draw the graph of $y = 2x - 1$ in the window [-7.9, 7.9] × [-2, 7]
   - b) Draw an epsilon-delta graph
   - c) Set ymin = 5 - .6 = 4.4 and ymax = 5 + .6 = 5.6.
   - d) Find xmin and xmax so that the graph goes corner to corner on the screen.
   - e) Determine the value of δ by looking at the window coordinates and the position of the graph.

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