

EULER'S FORMULA

A TI-83™ GRAPHING CALCULATOR ACTIVITY

Leonhard Euler – A Brief Biography

Leonhard Euler (1707-1783) has been called the greatest mathematician of the eighteenth century, and is one of the most prolific mathematicians of all time. His published works fill over 90 volumes. What really makes this remarkable is that for the last twenty years of his life, he was blind.

There are Euler formulas and numbers in mechanics, rotations, motion of fluids, and in the Calculus. Present-day Calculus students are most familiar with his formula that defines powers of imaginary numbers in terms of trigonometric functions. Euler also worked on problems that became seminal ones in Topology, notably the famous problem known as “The Seven Bridges of Konigsberg”

He worked in both Berlin and St. Petersburg, at the invitations of the reigning monarchs, Frederick the Great of Prussia, and Catherine the Great of Russia. He married twice, and had 13 children.

The contemporary French mathematician Condorcet tells the story of two of Euler's students who had independently added seventeen terms of a complicated infinite series, only to disagree in the fifteenth decimal place; Euler recomputed the sum mentally, thus settling the dispute.

Given that Euler made so many contributions to many different areas of mathematics, it is reasonable to wonder, “Which one of his formulas does this activity refer to?” The Euler’s formula that we will consider here relates four of the most important numbers in mathematics: e , i , and π , and -1 . As we will see, this particular Euler’s Formula is truly remarkable. It is both “beautiful” and the same time is one of the shortest ones on record.

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The following development of Euler’s Formula has been adapted from **Lectures on Physics**, chapter 22, by the Physicist and Nobel Prize winner Richard Feynman, and from a paper presented by James Lang at the 1998 T³ National Conference held in Nashville, TN.

Euler's Formula for $e^{i\pi}$ obviously forces us to consider the meaning of imaginary numbers when used as exponents. So the first question we will investigate is, "What is a *reasonable* meaning for using imaginary numbers as exponents?" In this activity, we will consider particularly what is meant by e^{ix} , where x is any real number and i is the imaginary unit: $i = \sqrt{-1}$.

Any meaning we apply to e^{ix} should at least satisfy the common laws of exponents. In particular, we would want

$$e^{i(x+y)} = e^{ix+iy} = e^{ix} e^{iy} \quad \text{and} \quad e^{0i} = e^0 = 1.$$

Then we can calculate $e^{.02i}$ if we know the value of $e^{.01i}$, since $e^{.01i}(e^{.01i}) = e^{(.01i+.01i)} = e^{.02i}$.

We can continue and multiply $e^{.02i}$ by $e^{.01i}$ to obtain $e^{.03i}$, etc.

First, we need a reasonable approximation for $e^{.01i}$. To establish such a reasonable value, use the calculator to complete the table below:

x	e^x	x	e^x	x	e^x
.1		.0001		.0027	
.05		.00005		.0073	
.01		.00001		.000251	
.005		.000005		.0000319	
.001		.000001		.0000057	

TABLE 1

What does this suggest as a formula for the value of e^x in terms of x for values of x close to zero? Pay attention to the significant digits when formulating your answer.

1. $e^x \approx$ _____

Extending your answer to the previous question to include complex exponents, it (hopefully) seems reasonable to take

$$\underline{e^{ix} \approx 1 + ix} \quad \text{for values of } x \text{ close to zero.}$$

$$\text{Therefore, } \underline{e^{.01i} \approx 1 + .01i}$$

Using this formula, here are two additional powers of e :

$$\begin{aligned} e^{.02i} &= e^{.01i+.01i} \\ &= e^{.01i} \cdot e^{.01i} \\ &\approx (1 + .01i)(1 + .01i) \\ &= 1 + .02i + .0001i^2 \\ &= 1 + .02i - .0001 \\ &= .9999 + .02i \\ &\approx 1.00 + .02i \end{aligned}$$

$$\begin{aligned} e^{.03i} &= e^{.02i+.01i} \\ &= e^{.02i} \cdot e^{.01i} \\ &\approx (.9999 + .02i)(1 + .01i) \\ &= .9999 + .009999i + .02i + .0002i^2 \\ &= .9999 + .029999i - .0002 \\ &= .9997 + .029999i \\ &\approx 1.00 + .03i \end{aligned}$$

You calculate the values in the table below. Use the full decimal binomial expressions on the next-to-last line of the previous page for $e^{.02i}$ ($\approx .9999 + .02i$) and $e^{.03i}$ ($\approx .9997 + .029999i$). Make use of the TI-83's ability to do complex arithmetic. Then write your answers here to the nearest hundredth.

$e^{.04i} =$	$e^{.05i} =$
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Since finding these is a tedious task for higher powers, we will use a computer program to do the calculations for us.

The following program will generate powers of $e^{.01i}$. It stores the power of $(e^{.01i})$ ($e^{.01i} \approx 1 + .01i$) in list L₁, the value of the power in list L₂, the real part of the power in list L₃ and the imaginary part of the power in list L₄.

Euler Formula Program

NOTES and COMMENTS

PROGRAM EULER

```

:ClrHome
:ClrAllLists
:Fix 2
:1+.01i→E
:0→L1(1)
:1→L2(1)
:1→L3(1)
:0→L4(1)
:20→F
:Lbl 1
:C/100→D
:Disp D,E
:C/F→G
:If int(G)≠G:Got
o 3
:G+1→L
:C/100→L1(L)
:E→L2(L)
:real(E)→L3(L)
:imag(E)→L4(L)
:Lbl 3
:(1+.01i)E→E
C+1→C
If C>650:Goto 2
Goto 1
Lbl 2
:End

```

Clear Home Screen
Clear lists – this is a memory intensive program
Display 2 places after the decimal point

Initialize variables and first elements in the lists.
(give starting values)

beginning of main processing loop

Save SELECTED values in table
(only every 0.2)

Multiply current value of
E by $(1 + .01i)$ to get next value of E.

Allow loop to repeat 650 times

End of main processing loop

LOOP

Run the program, and choose option 2: save every two-tenths (.2) . The program takes several minutes to run. Then use <STAT>-<EDIT> to examine the table in the calculator and complete the table below with *values to 2 decimal places*. (Move the cursor to the L₂ column. As you move the cursor down the column, copy the values from the *bottom line* of the TABLE screen.) The first two are done as examples.

x (L ₁)	e^{ix} (L ₂)	x (L ₁)	e^{ix} (L ₂)	x (L ₁)	e^{ix} (L ₂)	x (L ₁)	e^{ix} (L ₂)
0.0	1.00	1.8		3.6		5.4	
0.2	.98+.20i	2.0		3.8		5.6	
0.4		2.2		4.0		5.8	
0.6		2.4		4.2		6.0	
0.8		2.6		4.4		6.2	
1.0		2.8		4.6		6.4	
1.2		3.0		4.8			
1.4		3.2		5.0			
1.6		3.4		5.2			

TABLE 2

We could obtain still closer results by using $e^{.001i} \approx 1+.001i$. The program would then take considerably longer to run, because twenty times as many powers would be calculated. In order to do this, some steps in the program would have to be modified. What are these necessary changes?

Look at the real parts of the list L₂ in TABLE 2 above. Can you think of a function in which the values behave in approximately the same way? Remember that our original value for e^{0i} was an approximation, so the values in the table should not be interpreted as being exact, only close.

2. $y =$ _____

Look at the imaginary coefficients in TABLE 2 above. Can you think of a function whose values behave in approximately the same way?

3. $y =$ _____

Turn on Plot 1 (<StatPlot>) and draw a scatter plot, using Lists L₁ and L₃. Remember that L₃ is the list of real parts. Execute <Zoom> - 9 (ZOOMSTAT).

Show the scatter plot in the space:

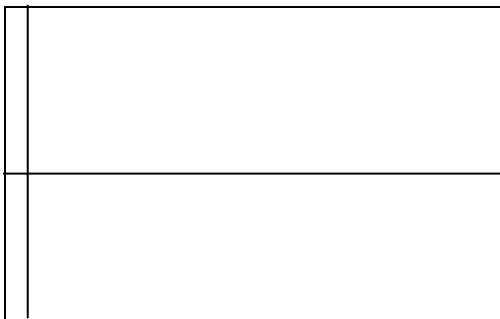


What does the graph look like? Write the equation of your conjecture.

4. $y =$ _____

Leave the stat plot turned on, and graph $y1 = \cos(x)$. Notice that the graph is virtually a perfect fit for the data. Draw $y = \cos x$ on the axes above.

Next, turn Plot 1 off (<StatPlot>), and use Plot 2 to create a scatter plot using lists L₁ and L₄. Remember that L₄ is the list of imaginary coefficients. Graph.



What does the graph look like? Write the equation of your conjecture. 5. $y =$ _____

Perform a **sin regression** on L₁, L₄, Y₁: <Stat>-<Calc>-<SinReg> L₁,L₄,Y₁. The sin regression will not generate any **r** values, so your judgment as to the “goodness” of the fit will be by eye, only. This regression will take a while. Superimpose the graph of the regression equation on the scatter plot above

Notice how closely the plot of the imaginary part does resemble the curve you guessed in the previous question.

Write the regression equation you get with a , b , c , and d **rounded to the nearest tenth**:

6. $y =$ _____

Re-write this equation in simplest form: 7. $y =$ _____

Recall that the real part of the imaginary number $a + bi$ is a , and the imaginary part is bi . Considering the small error introduced by the approximation of $e^{0.1i} \approx 1+.01i$ what conclusion can you *reasonably* draw from your work up to this point about the behavior of the real and imaginary parts of the powers of e^{xi} ?

8. Real part = _____ Imaginary part = _____

Therefore,

9. $e^{ix} =$ _____ **(1)**

Substitute π for x in equation **(1)**, and simplify. You will need to use the values of sin π and cos π .

$e^{i\pi} =$ _____

This gives us:

Euler’s Formula: $e^{i\pi} =$ _____

ANSWER SHEET

NAME _____

x	e^x	x	e^x	x	e^x
.1		.0001		.0027	
.05		.00005		.0073	
.01		.00001		.000251	
.005		.000005		.0000319	
.001		.000001		.0000057	

TABLE 1

Page 3

Page 4 table:

1. $e^x \approx$ _____

$e^{.04i} =$	$e^{.05i} =$
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Page 5 table

X (L1)	e^{ix} (L2)	X (L1)	e^{ix} (L2)	X (L1)	e^{ix} (L2)	X (L1)	e^{ix} (L2)
0.0	<u>1.00</u>	1.8		3.6		5.4	
0.2	<u>.98+.20i</u>	2.0		3.8		5.6	
0.4		2.2		4.0		5.8	
0.6		2.4		4.2		6.0	
0.8		2.6		4.4		6.2	
1.0		2.8		4.6		6.4	
1.2		3.0		4.8			
1.4		3.2		5.0			
1.6		3.4		5.2			

TABLE 2

2. $y =$ _____

Scatter Plots and Graphs

3. $y =$ _____

After #3

4. $y =$ _____

Page 6

Before #5

5. $y =$ _____

6. $y =$ _____

7. $y =$ _____

8. Real part = _____ Imaginary part = _____

9. $e^{ix} =$ _____ (1)

Euler's Formula: $e^{i\pi} =$ _____

10.

$$e^{iA} e^{iB} = (\cos A + i \sin A)(\cos B + i \sin B)$$

$$= \text{-----} \quad (\text{multiply out})$$

$$= \text{-----} \quad (\text{simplify using } i^2 = -1)$$

$$e^{iA} e^{iB} = (\text{-----}) + i(\text{-----}) \quad (\text{group terms and factor out } i) \quad \mathbf{(2)}$$

$$11. e^{i(A+B)} = \text{-----} \quad \mathbf{(3)}$$

$$12. \text{-----} \quad \mathbf{(4)}$$

$$13. \text{-----} = \text{-----} \qquad 14. \text{-----} = \text{-----}$$
